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2009 J. Phys. A: Math. Theor. 42 254003

(http://iopscience.iop.org/1751-8121/42/25/254003)

J. Phys. A: Math. Theor. **42** (2009) 254003 (121pp) doi:10.1088/1751-8113/42/25/254003

Foundations of the $AdS_5 \times S^5$ superstring: I

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Received 30 January 2009, in final form 18 April 2009
Published 9 June 2009
Online at stacks.iop.org/JPhysA/42/254003

Abstract
We review the recent advances towards finding the spectrum of the $AdS_5 \times S^5$ superstring. We thoroughly explain the theoretical techniques which should be useful for the ultimate solution of the spectral problem. In certain cases our exposition is original and cannot be found in the existing literature. The present part I deals with foundations of classical string theory in $AdS_5 \times S^5$, light-cone perturbative quantization and the derivation of the exact light-cone world-sheet scattering matrix.

PACS numbers: 11.25.Tq, 11.55.Ds

(Some figures in this article are in colour only in the electronic version)

Contents

Introduction

In the mid 1970s it became clear that the existing theoretical tools are hardly capable of providing an ultimate solution to the theory of strong interactions–quantum chromodynamics (QCD). At small distances quarks interact weakly and the physical properties of the theory can be well described by perturbative expansion based on Feynman diagrammatics. However, at large separation, forces between quarks become strong and this precludes the usage of perturbation theory. Understanding the strong coupling dynamics of quantum Yang–Mills theories remains one of the daunting challenges of theoretical particle physics.

A spectacular new insight into dynamics of non-Abelian gauge fields has recently been offered by the AdS/CFT (Anti-de-Sitter/conformal field theory) duality conjecture also known under the name of the 'gauge-string correspondence' [1]. This conjecture states that certain four-dimensional quantum gauge theories could be alternatively described in terms of closed strings moving in a ten-dimensional curved spacetime.

The prime example of the gauge-string correspondence involves the four-dimensional maximally supersymmetric $\mathcal{N} = 4$ Yang–Mills theory with gauge group $\mathrm{SU}(N)$ and type IIB superstring theory defined in an $\mathrm{AdS}_5 \times \mathrm{S}^5$ spacetime, which is the product of a five-dimensional Anti-de-Sitter space (the maximally symmetric space of constant negative curvature) and a 5-sphere. Since no candidate for a string dual of QCD is presently known, the $\mathcal{N} = 4$ theory together with its conjectured string partner offers a unique playground for testing the correspondence between strings and quantum field theories, as well as for understanding strongly coupled gauge theories in general. The success of the whole gauge-string duality program relies on our ability to quantitatively verify this prime example of the correspondence and, more importantly, to clarify the physical principles at work.

The $\mathcal{N} = 4$ super Yang–Mills theory has a vanishing beta-function and, for this reason, is an exact conformal field theory in four dimensions. The algebra of conformal transformations coincides with $\mathfrak{so}(4, 2)$ which, in addition to the Poincaré algebra, includes the generators of scale transformations (dilatation) and conformal boosts. The supersymmetry generators extend the conformal algebra to the superconformal algebra $\mathfrak{psu}(2, 2|4)$, the latter being the full algebra of global symmetries of the $\mathcal{N} = 4$ theory. Simultaneously, $\mathfrak{psu}(2, 2|4)$ plays the role of the symmetry algebra of type IIB superstring in the $\mathrm{AdS}_5 \times \mathrm{S}^5$ background. Thus, the gauge and string theory at hand share the same kinematical symmetry. This, however, does not *a priori* imply their undoubted equivalence.

To solve a conformal field theory, one has to identify the spectrum of primary operators (forming irreducible representations of the conformal group) and to compute their three-point correlation functions. Scaling (conformal) dimensions of primary operators and the three-point correlators encode all the information about the theory since all higher point correlation

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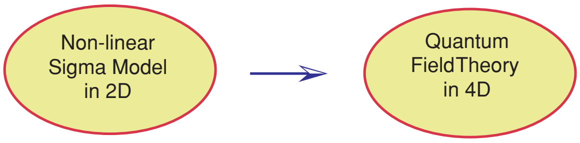

Figure 1. The AdS/CFT correspondence: the spectrum of a 2D nonlinear sigma-model describing string theory on a curved background is expected to be equivalent to the spectrum of a 4D quantum non-Abelian gauge theory in the large N limit.

functions can in principle be found by using the Operator Product Expansion. The $\mathcal{N} = 4$ theory has two parameters: the coupling constant $g_{\rm YM}$ and the rank N of the gauge group, and it admits a well-defined 't Hooft expansion in powers of $1/N$ with the 't Hooft coupling $\lambda = g_{\rm YM}^2 N$ kept fixed. The AdS/CFT duality conjecture relates these parameters to the string coupling constant g_s and the string tension g as follows: $g_s = \lambda/4\pi N$ and $g = \sqrt{\lambda}/2\pi$. Scaling dimensions Δ of composite gauge invariant primary operators are eigenvalues of the dilatation operator and they depend on the couplings: $\Delta \equiv \Delta(\lambda, 1/N)$. Scaling dimension is the only label of a (super)-conformal representation which is allowed to continuously depend on the parameters of the model. In spite of the finiteness of the $\mathcal{N} = 4$ theory, composite operators undergo non-trivial renormalization which explains the appearance of coupling-dependent anomalous dimensions. Alternatively, in string theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$ energies E of string states are functions of the couplings: $E \equiv E(g, g_s)$. In the most general setting, the gauge-string duality conjecture implies that physical states of gauge and string theories are organized in precisely the same set of $\mathfrak{psu}(2, 2|4)$-multiplets. In particular, energies of string states measured in the global AdS coordinates must coincide with scaling dimensions of gauge theory primary operators, both regarded as non-trivial functions of their couplings. Exhibiting this fact would be the first important step toward proving the conjecture.

The initial research on the $\mathcal{N} = 4$ gauge-string duality was concentrated on deriving scaling dimensions/correlation functions of primary operators in the supergravity approximation [2, 3]. This corresponds to the strongly coupled planar regime in the gauge theory where λ is infinite and N is large. Only rather special states—those which are protected from renormalization by a large amount of supersymmetry—could be a subject of comparison here.

The next important step has been undertaken in [4], where a special scaling limit was introduced. This work initiated intensive studies of unprotected operators with large R-charge which eventually led to the discovery of integrable structures in the gauge theory [5–7]. This discovery marked a new phase in the research on the fundamental model of AdS/CFT.

In the limit where the rank of the gauge group becomes infinite, one can neglect string interactions and consider free string theory. Free strings propagating in a non-trivial gravitational background such as $\mathrm{AdS}_5 \times \mathrm{S}^5$ are described by a two-dimensional quantum nonlinear sigma model. Finding the spectrum of the sigma model will determine the spectrum of scaling dimensions of composite operators in the dual gauge theory, figure 1.

In general, to solve a nonlinear quantum sigma model would be a hopeless enterprise. Remarkably, it appears, however, that classical strings in $\mathrm{AdS}_5 \times \mathrm{S}^5$ are described by an integrable model [8]. Integrable models constitute a special class of dynamical systems with an infinite number of conservation laws which in many cases hold the key to their exact solution. If string integrability continues to exist for the corresponding quantum theory then we are facing a breathtaking possibility of solving the string model exactly and, via the gauge-string duality, to find an exact solution of an interacting quantum field theory in four dimensions.

In recent years there has been a lot of exciting progress towards understanding integrable properties of both the string sigma model and the dual gauge theory. Not all this progress is yet logically deducible from the first principles and in certain cases it is based on new assumptions or clever guesses. Nevertheless, we feel that a clear and self-contained picture starts to emerge of how to obtain a solution (spectrum) of quantum strings in $AdS_5 \times S^5$. It is the scope of this review to explain this picture and to provide all the necessary technical tools in its support.

The review should be accessible to PhD students. It is certainly desirable to have a prerequisite knowledge of string theory [9]. The review might also be useful for specialists: as a handbook and as a source of formulae. In order not to distract the reader's attention with references, we comment on the literature in a special section concluding each section. Further, we emphasize that this review is most exclusively about string theory. To get more familiar with gauge theory constructions, the reader is invited to consult the original literature and reviews [10, 11].

As is seen for the moment, solving the string sigma-model is a complicated multi-step procedure. In view of this, before we start our actual journey, we would like to briefly describe the corresponding steps and to summarize the most relevant current progress in the field. This will also help the reader to become familiar with the content of the review.

Light-cone gauge. The starting point is the Green–Schwarz action for strings in $AdS_5 \times S^5$ which defines a two-dimensional nonlinear sigma model of Wess–Zumino type [12]. The isometries of the $AdS_5 \times S^5$ spacetime constitute the global symmetry algebra of the sigma model and string states are naturally characterized by the charges (representation labels) they carry under this symmetry algebra. Among all representation labels two charges, J and E, are of particular importance for the light-cone gauge fixing. The charge J is the angular momentum carried by the string due to its rotation around the equator of S^5 and E is the string energy, the latter corresponds to the symmetry of the Green–Schwarz action under constant shifts of the global time coordinate of the AdS space. It is the energy spectrum of string states that we would like to determine and subsequently compare to the spectrum of scaling dimensions of primary operators in the gauge theory.

To describe the physical states, it is advantageous to fix the so-called generalized light-cone gauge. In this gauge the world-sheet Hamiltonian is equal to $E - J$, while the light-cone momentum P_+ is another global charge which, generically, is a linear combination of J and E. Physical states should satisfy the level-matching condition: the total world-sheet momentum carried by a state must vanish. Solving the model is then equivalent to computing the physical spectrum of the (quantized) light-cone Hamiltonian for a fixed value of P_+.

Fixing the light-cone gauge for the Green–Schwarz string in a curved background is subtle because of a local fermionic symmetry. This question has been studied in [13, 14] where the exact gauge-fixed classical Hamiltonian was found. This Hamiltonian is non-polynomial in the world-sheet fields and, as such, can hardly be quantized in a straightforward manner.

From cylinder to plane: decompactification and symmetries. In the light-cone gauge the world-sheet action depends explicitly on the light-cone momentum P_+. By appropriately rescaling the world-sheet coordinates, the theory becomes defined on a cylinder of circumference P_+. At this stage, one can consider the decompactification limit, i.e. the limit where P_+ and therefore the radius of the cylinder go to infinity, while keeping the string tension fixed. In this limit one is left with a theory on a plane which leads to significant simplifications. Most importantly, the world-sheet theory has a massive spectrum and the notion of asymptotic states (particles) is well defined, calling for an application of scattering theory. Quantum integrability should then imply the absence of particle production and factorization of multi-particle scattering into a sequence of two-body events.

Thus, assuming quantum integrability, the next step is to find the dispersion relation for elementary excitations and the *S*-matrix describing their pairwise scattering. To deal with particles with arbitrary world-sheet momenta, one has to give up the level-matching condition. This leads to an important modification of the global symmetry algebra of the model. Namely, the manifest $\mathfrak{psu}(2|2) \oplus \mathfrak{psu}(2|2) \subset \mathfrak{psu}(2,2|4)$ symmetry algebra of the light-cone string theory gets enhanced by two central charges [15]. The central charges vanish on physical states satisfying the level-matching condition but they play a crucial role in fixing the structure of the world-sheet *S*-matrix. The same centrally extended algebra also appears in the dual gauge theory [16].

Dispersion relation and scattering matrix. Insights coming from both gauge and string theory [4] led to a conjecture for the dispersion relation [17]. It has the following unusual form:

$$\epsilon(p) = \sqrt{1 + 4g^2 \sin^2 \frac{p}{2}},$$

where g is the string tension, ϵ and p are the energy and the momentum of an elementary excitation.

An important observation made in [16] is that the dispersion relation is uniquely determined by the symmetry algebra of the model provided its central charges are known as exact functions of the string tension and the world-sheet momentum. The dispersion relation is non-relativistic although it reveals the usual square root dependence of relativistic field theory. On the other hand, the sine function under the square root is a common feature of lattice theories, and its appearance here is rather surprising, given that the string world-sheet is continuous.

The various pieces of the two-body scattering matrix were conjectured in [18, 19, 21] based on the analysis of the integral equations [22] describing classical spinning strings [23–25] and insights from gauge theory [17]. Later, it was found that the matrix structure of this *S*-matrix is uniquely fixed by the centrally extended $\mathfrak{psu}(2|2)\oplus\mathfrak{psu}(2|2)$ symmetry algebra, the Yang–Baxter equation and the generalized physical unitarity condition [16, 26, 27].

Dressing factor. The *S*-matrix is thus determined up to an overall scalar function $\sigma(p_1, p_2)$—the so-called dressing factor [18]. Ideally, one would hope that further physical requirements would allow for complete determination of this factor. In relativistic integrable quantum field theories implementation of Lorentz invariance together with crossing symmetry exchanging particles with anti-particles imposes an additional crossing relation on the *S*-matrix [28].

The light-cone gauge-fixed sigma model is not Lorentz-invariant. However, as was argued in [29], some version of the crossing relation might hold for the corresponding *S*-matrix; the crossing relation then implies a non-trivial functional equation for the dressing factor. This crossing equation is rather complicated; it is unclear how to solve it in full generality and how to single out the physically relevant solution.

Luckily, the logarithm of the dressing factor turns out to be a 2-form on the vector space of local conserved charges of the model which severely constraints its functional form [18]. The dressing factor explicitly depends on the string tension g and admits a ‘strong coupling’ expansion in powers of $1/g$ that corresponds to an asymptotic perturbative expansion of the string sigma model.

Combining the functional form of the dressing factor together with the first two known orders in the strong coupling expansion [18, 30], a set of solutions to the crossing equation in terms of an all-order strong coupling asymptotic series has been proposed [31]. A particular solution was conjectured to correspond to the actual string sigma model perturbative expansion. This solution was shown to agree with the explicit two-loop sigma model result [32, 33]. It should be stressed, however, that all these solutions are only asymptotic and, therefore, they do not define the dressing factor as a function of g.

In contrast to the strong coupling expansion, gauge theory perturbative expansion of the dressing factor is in powers of g and it has a finite radius of convergence. As a result, the dressing factor can be defined as a function of g. An interesting proposal for the exact dressing factor has been put forward in [34]. On the one hand, it agrees with the explicit four-loop gauge theory computation [35, 36]. On the other hand, it was argued to have the same strong coupling asymptotic expansion as the particular solution by [31] corresponding to the string sigma model. Taking all this into account, one can adopt the working assumption that the exact dressing factor and, therefore, the S-matrix are established. However, a word of caution to bear in mind—there is no unique solution to the crossing equation; additional yet to be found physical constraints should be used to single out the right solution unambiguously.

Bound states. Having found the exact dispersion relation and the S-matrix, the next step is to determine the complete asymptotic spectrum of the model. This amounts to finding all bound states of the elementary excitations and bound states of the bound states, etc. This problem can be solved by analyzing the pole structure of the S-matrix. The analysis reveals that all bound states are those of elementary particles [37]. More explicitly, Q-particle bound states comprise into the tensor product of two $4Q$-dim atypical totally symmetric multiplets of the centrally extended symmetry algebra $\mathfrak{su}(2|2)$ [38]. Since the light-cone string sigma model is not Lorentz-invariant, the identification of what is called the 'physical region' of the S-matrix is very subtle and it affects the counting of bound states [27].

The problem of computing a bound state S-matrix is rather non-trivial and reduces to finding its dressing factor and fixing its matrix structure. The dressing factor can be computed by using the fusion procedure for the $\mathfrak{su}(2)$ sector S-matrix [39, 40], and appears to be of the same universal form as that for the elementary particles S-matrix [18]. As to the matrix structure, it can be found by using the superfield approach by [41].

Back from plane to cylinder: finite P_+ spectrum. Having understood the spectrum of the light-cone string sigma model on a plane, one has to 'upgrade' the findings to a cylinder. All physical string configurations (and dual gauge theory operators) are characterized by a finite value of P_+, and as such they are excitations of a theory on a cylinder.

The first step in determining the finite-size spectrum of a two-dimensional integrable model is to consider the model on a cylinder of a very large but finite circumference P_+. In this case integrability implies that a multi-particle state can be approximately described by the wavefunction of the Bethe-type [28]. Factorizability of the multi-particle scattering matrix together with the periodicity condition for the Bethe wavefunction leads to a system of equations on the particle momenta known as the Bethe–Yang equations. In the AdS/CFT context these equations are usually referred to as the asymptotic Bethe ansatz[4] [21]. The

[4] In the theory of integrable models the asymptotic Bethe ansatz has been known for a long time [20].

$AdS_5 \times S^5$ string S-matrix has a complicated matrix structure which results in the end in a set of nested Bethe equations [16, 21, 42].

The Bethe–Yang equations determine any power-like $1/P_+$ corrections to the energy of multi-particle states. It is known, however, that for large P_+ there are also exponentially small corrections. To compute the leading exponential corrections, one can adapt Lüscher's formulae [43, 44] for the non-Lorentz-invariant case at hand [45]. This computation has been done for some string states at strong coupling.

Remarkably, Lüscher's approach could also be applied to find perturbative scaling dimensions of gauge theory operators up to the first order where the Bethe–Yang description breaks down [46]. The corresponding computation has been done [46] for the simplest case of the so-called Konishi operator and stunning agreement with a very complicated four-loop result based on the standard Feynman diagrammatics [47] has been found. String theory starts to reveal its extreme power, elegance and simplicity in comparison to the conventional perturbative approach!

Thermodynamic Bethe ansatz. The success in computing gauge theory perturbative anomalous dimensions is very encouraging. However, one is really interested in non-perturbative gauge theory, i.e. in the exact spectrum for finite values of the gauge coupling (or equivalently for finite string tension and finite P_+). One tempting possibility is to generalize the thermodynamic Bethe ansatz (TBA), originally developed for relativistic integrable models [48], to the light-cone string theory at hand.

The TBA approach would be based on the following construction. Consider a closed string of length $L \equiv P_+$ which wraps a loop of 'time' length R. The topology of the corresponding surface spanned by the string is a torus, i.e. the Cartesian product of two orthogonal circles with circumferences L and R , respectively. According to the imaginary time formalism of statistical mechanics, the circumference of any of these two circles can be treated as the inverse temperature for a statistical field theory with the Hilbert space of quantum-mechanical states defined on the complementary circle. Thus, there are two models related to one and the same torus: the original theory of strings with length L at temperature $1/R$ and the 'mirror' model defined on a circle of length R at temperature $1/L$. The smaller and the colder the original theory, the hotter and the bigger its mirror. In particular, the ground-state energy of the original string model in a finite one-dimensional volume L is equal to the Gibbs free energy (or Witten's index in the case of periodic fermions) of the mirror model in infinite volume, i.e. for infinite R. It should be also possible to relate the whole string spectrum to the proper thermodynamic quantities of the mirror model defined for infinite R, a problem which is not well understood at present.

Since the light-cone string sigma model is not Lorentz-invariant, the mirror model is governed by a different Hamiltonian and therefore has very different dynamics. Thus, to implement the TBA approach one has to study the mirror theory in detail. The first step in this direction has been already done in [27], where the Bethe–Yang equations for the mirror model were derived. Another result obtained in [27] was a classification of the mirror bound states according to which they comprise the tensor product of two $4Q$-dim atypical totally anti-symmetric multiplets[5] of the centrally extended algebra $\mathfrak{su}(2|2)$. This observation was of crucial importance for the derivation [46] of the scaling dimension of the Konishi operator. We consider this derivation as prime evidence for the validity of the mirror theory approach. Recently two interesting conjectures has been made: one concerns the classification of states

[5] Note the difference with the bound states in the original model which transform in symmetric representations!

contributing in the thermodynamic limit of the mirror theory [49], another formulates the so-called Y-system [50, 51] which is supposed to encode the finite-size string spectrum [52].

Because of a large amount of necessary material, we decided to split the review into two parts. The present part I deals with foundations of classical string theory in $AdS_5 \times S^5$, the light-cone perturbative quantization and derivation of the light-cone world-sheet scattering matrix. Part II will include the derivation of the Bethe–Yang equations, the discussion of bound states and the progress in understanding the finite-size spectrum of the string sigma model, both in Lüscher's and in the TBA setting. We will also present yet 'phenomenological' arguments which led to the determination of the dressing phase. In the last section of part II we plan to list the important topics which were uncovered in the present review.

This concludes our brief description of a possible approach to find the spectrum of quantum strings in $AdS_5 \times S^5$. At present we do not know if the route we follow is the unique or the simplest one. Time will tell. In any case, the success we encounter underway makes us believe that the first ever exact solution of a four-dimensional interacting quantum field theory is within our reach.

1. String sigma model

In addition to the flat ten-dimensional Minkowski space, type IIB supergravity admits another maximally supersymmetric solution which is product of the five-dimensional Anti-de-Sitter space AdS_5 and the 5-sphere S^5. This solution is supported by the self-dual Ramond–Ramond 5-form flux. The presence of this background flux precludes the usage of the standard NSR approach to build up the action for strings propagating in this geometry. Indeed, the Ramond–Ramond vertex operator is known to be non-local in terms of the world-sheet fields and, for this reason, it is unclear how to couple it to the string world-sheet.

There exists another approach to define string theory for a background geometry supported by Ramond–Ramond fields—the so-called Green–Schwarz formalism. This formalism has a further advantage, namely, it allows one to realize the spacetime supersymmetry in a manifest way. The Green–Schwarz approach can be used for any background obeying the supergravity equations of motion to guarantee the invariance of the corresponding string action with respect to the local fermionic symmetry (κ-symmetry), the latter being responsible for the spacetime supersymmetry of the physical spectrum. In practice, construction of the Green–Schwarz action for an arbitrary supergravity solution faces a serious difficulty. Namely, starting from a given bosonic solution, one has to determine the full structure of the type IIB superfield, a problem that has not been solved so far for a generic background.

Fortunately, there is an alternative approach to define the Green–Schwarz superstring which makes use of the special symmetry properties of the background solution. This approach has already been shown to work nicely in the case of a flat background, where it amounts to defining the Green–Schwarz string as a WZNW-type nonlinear sigma model on the coset superspace being a quotient of the ten-dimensional super-Poincaré group over its Lorentz subgroup SO(9, 1). The super-Poincaré group acts naturally on this coset space and it is a manifest symmetry of the corresponding sigma model action. The Wess–Zumino term guarantees invariance of the full action under κ-symmetry transformations.

Remarkably, a similar sigma model approach can be developed in the $AdS_5 \times S^5$ case. Namely, we define type IIB Green–Schwarz superstring in the $AdS_5 \times S^5$ background as a nonlinear sigma-model with target space being the following coset:

$$\frac{\mathrm{PSU}(2,2|4)}{\mathrm{SO}(4,1)\times \mathrm{SO}(5)}. \tag{1.1}$$

The supergroup PSU(2, 2|4) contains the bosonic subgroup SU(2, 2) × SU(4) which is locally isomorphic to SO(4, 2) × SO(6); the quotient of the latter over SO(4, 1) × SO(5) provides a model of the $AdS_5 \times S^5$ manifold with SO(4, 1) × SO(5) being the group of local Lorentz transformations. Correspondingly, the coset (1.1) can be regarded as a model of the $AdS_5 \times S^5$ superspace. The group PSU(2, 2|4) which acts on the coset by left multiplications plays the role of the isometry group of the $AdS_5 \times S^5$ superspace. Thus, considering a nonlinear sigma-model with target superspace (1.1) provides a natural way to couple the string world-sheet to the background Ramond–Ramond fields.

In this section we will describe the corresponding sigma-model in detail. We will discuss its global and local symmetries and show that it can be embedded into the standard framework of classical integrable systems.

1.1. Superconformal algebra

The construction of the coset sigma-model essentially relies on the properties of the superconformal algebra $\mathfrak{psu}(2,2|4)$. Here we will summarize the necessary facts about this algebra and introduce our notation.

1.1.1. Matrix realization of $\mathfrak{su}(2,2|4)$. We start our discussion with the definition of the superalgebra $\mathfrak{sl}(4|4)$ considered over the field $\mathbb{C}$. As a matrix superalgebra, $\mathfrak{sl}(4|4)$ is spanned by 8 × 8 matrices M, which we write in terms of 4 × 4 blocks as

$$M = \begin{pmatrix} m & \theta \\ \eta & n \end{pmatrix}. \tag{1.2}$$

These matrices are required to have vanishing supertrace $\mathrm{str}M \equiv \mathrm{tr}m - \mathrm{tr}n = 0$. The superalgebra $\mathfrak{sl}(4|4)$ carries the structure of a $\mathbb{Z}_2$-graded algebra: the matrices m and n are regarded as even, and θ, η as odd, respectively. The entries of θ and η can be thought of as grassmann (fermionic) anti-commuting variables.

The superalgebra $\mathfrak{su}(2,2|4)$ is a non-compact real form of $\mathfrak{sl}(4|4)$. It is identified with a set of fixed points $M^\star = M$ of $\mathfrak{sl}(4|4)$ under the Cartan involution[6] $M^\star = -HM^\dagger H^{-1}$. In other words, a matrix M from $\mathfrak{su}(2,2|4)$ is subject to the following reality condition:

$$M^\dagger H + HM = 0. \tag{1.3}$$

Here the adjoint of the supermatrix M is defined as $M^\dagger = (M^t)^*$ and the Hermitian matrix H is taken to be

$$H = \begin{pmatrix} \Sigma & 0 \\ 0 & \mathbb{1}_4 \end{pmatrix}, \tag{1.4}$$

where Σ is the following 4 × 4 matrix

$$\Sigma = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix} \tag{1.5}$$

and $\mathbb{1}_n$ denotes the $n \times n$ identity matrix. We further note that for any odd element θ the conjugation acts as a $\mathbb{C}$-anti-linear anti-involution

$$(c\theta)^* = \bar{c}\theta^*, \qquad \theta^{**} = \theta, \qquad (\theta_1\theta_2)^* = \theta_2^*\theta_1^*,$$

which guarantees, in particular, that $(M_1M_2)^\dagger = M_2^\dagger M_1^\dagger$, i.e. that anti-Hermitian supermatrices form a Lie superalgebra.

[6] It is worthwile to note that our definition of the Cartan involution is different but equivalent to the standard one: $M^\star = -\mathrm{i}^{\epsilon_M} HM^\dagger H^{-1}$, where $\epsilon_M = 0$ for even and $\epsilon_M = 1$ for odd elements, respectively.

Condition (1.3) implies that

$$m^\dagger = -\Sigma m \Sigma, \qquad n^\dagger = -n, \qquad \eta^\dagger = -\Sigma\theta. \tag{1.6}$$

Thus, m and n span the unitary subalgebras $\mathfrak{u}(2,2)$ and $\mathfrak{u}(4)$, respectively. The algebra $\mathfrak{su}(2,2|4)$ also contains the $\mathfrak{u}(1)$-generator $\mathrm{i}\mathbb{1}$, as the latter obeys equation (1.3) and has vanishing supertrace. Thus, the bosonic subalgebra of $\mathfrak{su}(2,2|4)$ is

$$\mathfrak{su}(2,2)\oplus\mathfrak{su}(4)\oplus\mathfrak{u}(1). \tag{1.7}$$

The superalgebra $\mathfrak{psu}(2,2|4)$ is defined as a *quotient algebra* of $\mathfrak{su}(2,2|4)$ over this $\mathfrak{u}(1)$-factor. It is important to note that $\mathfrak{psu}(2,2|4)$, as the quotient algebra, has no realization in terms of 8×8 supermatrices.

It is convenient to fix a basis for the bosonic subalgebra $\mathfrak{su}(2,2)\oplus\mathfrak{su}(4)$. Throughout this work we will use the following representation of Dirac matrices

$$\gamma^1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \qquad \gamma^2 = \begin{pmatrix} 0 & 0 & 0 & \mathrm{i} \\ 0 & 0 & \mathrm{i} & 0 \\ 0 & -\mathrm{i} & 0 & 0 \\ -\mathrm{i} & 0 & 0 & 0 \end{pmatrix}, \qquad \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$\gamma^4 = \begin{pmatrix} 0 & 0 & -\mathrm{i} & 0 \\ 0 & 0 & 0 & \mathrm{i} \\ \mathrm{i} & 0 & 0 & 0 \\ 0 & -\mathrm{i} & 0 & 0 \end{pmatrix}, \qquad \gamma^5 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \Sigma,$$

satisfying the SO(5) Clifford algebra relations

$$\gamma^i\gamma^j + \gamma^j\gamma^i = 2\delta^{ij}, \qquad i,j = 1,\ldots,5.$$

Note that $\gamma^5 = -\gamma^1\gamma^2\gamma^3\gamma^4$. All these matrices are Hermitian: $(\gamma^i)^* = (\gamma^i)^t$, so that $\mathrm{i}\gamma^i$ belongs to $\mathfrak{su}(4)$. The spinor representation of $\mathfrak{so}(5)$ is spanned by the generators $n^{ij} = \frac{1}{4}[\gamma^i,\gamma^j]$ satisfying the relations

$$[n^{ij}, n^{kl}] = \delta^{jk}n^{il} - \delta^{ik}n^{jl} - \delta^{jl}n^{ik} + \delta^{il}n^{jk}, \qquad n^{ij} = -n^{ji}. \tag{1.8}$$

Adding $n^{i6} = \frac{\mathrm{i}}{2}\gamma^i$, one can verify that $n^{ij} = -n^{ji}$ generate an irreducible (Weyl) spinor representation of $\mathfrak{so}(6)\sim\mathfrak{su}(4)$ with defining relations (1.8) where now $i,j = 1,\ldots,6$. The other Weyl representation would correspond to choosing $n^{i6} = -\frac{\mathrm{i}}{2}\gamma^i$.

Analogously, a set $\{\mathrm{i}\gamma^5, \gamma^i\}$ with $i = 1,\ldots,4$ generates the Clifford algebra for SO(4,1). Indeed, if we introduce $\gamma^0 \equiv \mathrm{i}\gamma^5$, then $m^{ij} = \frac{1}{4}[\gamma^i,\gamma^j]$ with $i,j = 0,\ldots,4$ satisfy the $\mathfrak{so}(4,1)$ algebra relations

$$[m^{ij}, m^{kl}] = \eta^{jk}m^{il} - \eta^{ik}m^{jl} - \eta^{jl}m^{ik} + \eta^{il}m^{jk}, \qquad m^{ij} = -m^{ji}, \tag{1.9}$$

where $\eta = \mathrm{diag}(-1,1,1,1,1)$. Enlarging this set of generators by $m^{i5} = \frac{1}{2}\gamma^i$, $i = 0,\ldots,4$, we obtain a realization of $\mathfrak{so}(4,2)\sim\mathfrak{su}(2,2)$ with the same defining relations (1.9) where this time $\eta = \mathrm{diag}(-1,1,1,1,1,-1)$ and $i,j = 0,\ldots,5$.

Thus, we regard $\mathfrak{su}(2,2)$ and $\mathfrak{su}(4)$ as real vector spaces spanned by the following set of generators:

$$\begin{aligned} &\mathfrak{su}(2,2)\sim\mathrm{span}_{\mathbb{R}}\left\{\frac{1}{2}\gamma^i, \frac{\mathrm{i}}{2}\gamma^5, \frac{1}{4}[\gamma^i,\gamma^j], \frac{\mathrm{i}}{4}[\gamma^5,\gamma^j]\right\}, \qquad i,j = 1,\ldots,4, \\ &\mathfrak{su}(4)\sim\mathrm{span}_{\mathbb{R}}\left\{\frac{\mathrm{i}}{2}\gamma^i, \frac{1}{4}[\gamma^i,\gamma^j]\right\}, \qquad i,j = 1,\ldots,5. \end{aligned} \tag{1.10}$$

Together with the central element $\mathrm{i}\mathbb{1}$, this set of generators provides an explicit basis for the bosonic subalgebra of $\mathfrak{su}(2,2|4)$.

Our next goal is to elaborate more on the structure of the conformal algebra $\mathfrak{su}(2,2)$. Introduce the notation $\gamma^{ij} = \frac{1}{4}[\gamma^i, \gamma^j]$. First, we note that the matrices $i\gamma^{15}, i\gamma^{25}, i\gamma^{35}, i\gamma^{45}$ together with $\gamma^{1,2,3,4}$ are block off-diagonal, i.e. in terms of 2×2 blocks they span the (real) eight-dimensional space

$$\begin{pmatrix} 0 & \bullet \\ \bullet & 0 \end{pmatrix} \subset \mathfrak{su}(2,2).$$

On the other hand, the matrices γ^{ij} with $i, j = 1, \ldots, 4$ span the $\mathfrak{so}(4)$ subalgebra embedded into the conformal algebra diagonally as two copies of $\mathfrak{su}(2)$

$$\begin{pmatrix} \mathfrak{su}(2) & 0 \\ 0 & \mathfrak{su}(2) \end{pmatrix} \subset \mathfrak{su}(2,2).$$

Finally, $\frac{i}{2}\gamma^5$ is diagonal and its centralizer in $\mathfrak{su}(2,2)$ coincides with the maximal compact subalgebra $\mathfrak{su}(2)\oplus\mathfrak{su}(2)\oplus\mathfrak{u}(1)\subset\mathfrak{su}(2,2)$. Sometimes the generator $\frac{1}{2}\gamma^5$ is referred to as the 'conformal Hamiltonian'.

Second, consider the one-dimensional subalgebra generated by $\frac{1}{2}\gamma^3 \equiv -\,i D$. It is usually called the 'dilatation subalgebra'. Evidently, in addition to γ^3, the centralizer of γ^3 in $\mathfrak{su}(2,2)$ is generated by $\gamma^{12}, \gamma^{14}, \gamma^{24}$ and $i\gamma^{15}, i\gamma^{25}, i\gamma^{45}$. The first three matrices generate $\mathfrak{so}(3)$, while, all together, the six matrices generate the Lorentz subalgebra $\mathfrak{so}(3,1)$. The orthogonal complement to $\mathfrak{so}(3,1)\oplus\, i D$ is the eight-dimensional real space. The basis in this space can be chosen from eigenvectors of $\,i D$. The eigenvectors K_i, $i = 1, \ldots, 4$ with negative eigenvalues form the subalgebra of *special conformal transformation*, while the eigenvectors P_i with positive eigenvalues form the subalgebra of *translations*.

Finally, we note that the matrices γ^3 and γ^5 are related by an orthogonal transformation

$$e^{-\frac{\pi}{4}\gamma^3\gamma^5}\gamma^3\, e^{+\frac{\pi}{4}\gamma^3\gamma^5} = \gamma^5 \tag{1.11}$$

implying thereby the well-known relation between the dilatation generator D and the conformal Hamiltonian. In unitary representations the operator D must be Hermitian: $D^\dagger = D$. Here $D = \frac{i}{2}\gamma^3$ is anti-Hermitian which is compatible with the fact that we are dealing with the finite-dimensional and, therefore, non-unitary representation of the non-compact algebra $\mathfrak{su}(2,2)$.

The following matrix K:

$$K = -\gamma^2\gamma^4 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{1.12}$$

will play a distinguished role in our subsequent discussion. One can check that for all Dirac matrices the following relation is satisfied:

$$(\gamma^i)^t = K\gamma^i K^{-1}, \qquad i = 1, \ldots, 5. \tag{1.13}$$

Also we define the charge conjugation matrix $C = \gamma^1\gamma^3$ which commutes with K and has the following properties:

$$C\gamma^i C^{-1} = -(\gamma^i)^t, \qquad C\gamma^5 C^{-1} = (\gamma^5)^t, \qquad C^2 = -\mathbb{1}, \quad i = 1, \ldots, 4.$$

1.1.2. $\mathbb{Z}_4$-grading. The outer automorphism group of a Lie algebra plays an important role in the corresponding representation theory. It appears that for $\mathfrak{sl}(4|4)$ the outer automorphism group $\mathrm{Out}(\mathfrak{sl}(4|4))$ contains continuous and finite subgroups.

Consider the continuous group $\{\delta_\rho, \rho \in \mathbb{C}^*\}$ which acts on M in the following way:

$$\delta_\rho(M) = \begin{pmatrix} m & \rho\theta \\ \frac{1}{\rho}\eta & n \end{pmatrix}, \tag{1.14}$$

i.e. it leaves the bosonic elements untouched and acts on the fermionic elements as a dilatation. In fact, this transformation is generated by the so-called hypercharge

$$\Upsilon = \begin{pmatrix} \mathbb{1}_4 & 0 \\ 0 & -\mathbb{1}_4 \end{pmatrix} \tag{1.15}$$

and can be formally written in the form $\delta_\rho(M) = e^{\frac{1}{2}\Upsilon \log\rho} M\, e^{-\frac{1}{2}\Upsilon \log\rho}$. Of course, the hypercharge is not an element of $\mathfrak{sl}(4|4)$ as it has non-vanishing supertrace. On the other hand,

$$e^{\frac{1}{2}\Upsilon \log\rho} = \begin{pmatrix} \rho^{\frac{1}{2}}\mathbb{1}_4 & 0 \\ 0 & \rho^{-\frac{1}{2}}\mathbb{1}_4 \end{pmatrix}. \tag{1.16}$$

The superdeterminant of this matrix is equal to ρ^4. Thus, for ρ satisfying the relation $\rho^4 = 1$, the corresponding automorphisms δ_ρ are, in fact, inner. Hence, the continuous family of outer automorphisms of $\mathfrak{sl}(4|4)$ coincides with the factor-group $\delta_\rho/\{\delta_\rho : \rho^4 = 1\}$. We further note that the automorphism group δ_ρ admits a restriction to $\mathfrak{su}(2,2|4)$ provided the parameter ρ lies on a circle $|\rho| = 1$.

The finite subgroup of $\mathrm{Out}(\mathfrak{sl}(4|4))$ coincides with the Klein four-group $\mathbb{Z}_2 \times \mathbb{Z}_2$. The first factor is generated by the transformation

$$M = \begin{pmatrix} m & \theta \\ \eta & n \end{pmatrix} \to \begin{pmatrix} n & \eta \\ \theta & m \end{pmatrix}, \tag{1.17}$$

while the second one is generated by

$$M \to -M^{\mathrm{st}}, \tag{1.18}$$

where the supertranspose M^{st} is defined as

$$M^{\mathrm{st}} = \begin{pmatrix} m^t & -\eta^t \\ \theta^t & n^t \end{pmatrix}. \tag{1.19}$$

The 'minus supertransposition' is an automorphism of order four. We see, however, that

$$(M^{\mathrm{st}})^{\mathrm{st}} = \begin{pmatrix} m & -\theta \\ -\eta & n \end{pmatrix} = \delta_{-1}(M), \tag{1.20}$$

which, according to the discussion above, is an inner automorphism. Thus, in the group of outer automorphisms the order of 'minus supertransposition' is indeed two, while in the group of all automorphisms its order is equal to four.

The fourth-order automorphism $M \to -M^{\mathrm{st}}$ allows one to endow $\mathfrak{sl}(4|4)$ with the structure of a $\mathbb{Z}_4$-graded Lie superalgebra. For our further purposes it is important, however, to choose an equivalent automorphism[7]

$$M \to \Omega(M) = -\mathcal{K} M^{\mathrm{st}} \mathcal{K}^{-1}, \tag{1.21}$$

where $\mathcal{K}$ is the 8×8-matrix, $\mathcal{K} = \mathrm{diag}(K, K)$, and the 4×4 matrix K is given in equation (1.12). On the product of two supermatrices one has $\Omega(M_1 M_2) = -\Omega(M_2)\Omega(M_1)$.

Introducing the notation $\mathscr{G} = \mathfrak{sl}(4|4)$, let us define

$$\mathscr{G}^{(k)} = \{M \in \mathscr{G}, \Omega(M) = i^k M\}. \tag{1.22}$$

[7] Although the actions of these two automorphisms are related by the similarity transformation, they introduce inequivalent $\mathbb{Z}_4$-graded structures on $\mathfrak{sl}(4|4)$.

Then, as a vector space, $\mathscr{G}$ can be decomposed into a direct sum of graded subspaces

$$\mathscr{G} = \mathscr{G}^{(0)} \oplus \mathscr{G}^{(1)} \oplus \mathscr{G}^{(2)} \oplus \mathscr{G}^{(3)} \tag{1.23}$$

where $[\mathscr{G}^{(k)}, \mathscr{G}^{(m)}] \subset \mathscr{G}^{(k+m)}$ modulo $\mathbb{Z}_4$. For any matrix $M \in \mathscr{G}$ its projection $M^{(k)} \in \mathscr{G}^{(k)}$ is given by

$$M^{(k)} = \tfrac{1}{4}(M + \mathrm{i}^{3k}\Omega(M) + \mathrm{i}^{2k}\Omega^2(M) + \mathrm{i}^{k}\Omega^3(M)). \tag{1.24}$$

It is easy to see that the projections $M^{(0)}$ and $M^{(2)}$ are even, while $M^{(1)}$ and $M^{(3)}$ are odd.

While $[K, \Sigma] = [\gamma^5, \gamma^2\gamma^4] = 0$, in general $(M^{\mathrm{st}})^\dagger \neq (M^\dagger)^{\mathrm{st}}$. As a result, one finds that the action of Ω (anti-) commutes with the Cartan involution:

$$\begin{aligned} \Omega(M)^\dagger &= \Omega(M^\dagger) \qquad \text{for} \quad M \text{ even}, \\ \Omega(M)^\dagger &= -\Omega(M^\dagger) \qquad \text{for} \quad M \text{ odd}. \end{aligned} \tag{1.25}$$

In fact, these two formulae can be concisely written as a single expression

$$\Omega(M)^\dagger = \Upsilon\Omega(M^\dagger)\Upsilon^{-1} = -(\Upsilon H)\Omega(M)(\Upsilon H)^{-1}, \tag{1.26}$$

where Υ is hypercharge (1.15) and we assumed that $M \in \mathfrak{su}(2,2|4)$. Thus, Ω admits a restriction to the bosonic subalgebra of the real form $\mathfrak{su}(2,2|4)$. On the whole $\mathfrak{su}(2,2|4)$ the map Ω is not diagonalizable, since two eigenvalues of Ω are imaginary: for the projections $M^{(k)}$ with $k = 1, 3$ we have $\Omega(M^{(k)}) = \pm\mathrm{i}M^{(k)}$, while $\mathfrak{su}(2,2|4)$ is a Lie superalgebra over real numbers. Nevertheless, any matrix $M \in \mathfrak{su}(2,2|4)$ can be uniquely decomposed into the sum (1.23), where each component $M^{(k)}$ takes values in $\mathfrak{su}(2,2|4)$. To make this point clear, we compute the Hermitian-conjugate of $M^{(k)}$ given by equation (1.24)

$$M^{(k)\dagger} = -\tfrac{1}{4}H[M + \mathrm{i}^{k}\Upsilon\Omega(M)\Upsilon^{-1} + \mathrm{i}^{2k}\Omega^2(M) + \mathrm{i}^{3k}\Upsilon\Omega^3(M)\Upsilon^{-1}]H^{-1}$$

where we made use of equations (1.26) and (1.3). It remains to note that according to equation (1.20) one has $\Upsilon\Omega(M)\Upsilon^{-1} = \Omega^3(M)$ so that $M^{(k)\dagger} = -HMH^{-1}$, i.e. $M^{(k)}$ belongs to $\mathfrak{su}(2,2|4)$ for any k. Thus, denoting now $\mathscr{G} = \mathfrak{su}(2,2|4)$, in what follows we will refer to equation (1.23) as the $\mathbb{Z}_4$-graded decomposition of $\mathfrak{su}(2,2|4)$, where the individual subspaces are defined by means of equation (1.24).

According to our discussion, with respect to the action of Ω the bosonic subalgebra $\mathfrak{su}(2,2) \oplus \mathfrak{su}(4) \oplus \mathfrak{u}(1) \subset \mathfrak{su}(2,2|4)$ is decomposed into the direct sum of two graded components. Working out explicitly the projection $M^{(0)}$, one finds

$$M^{(0)} = \frac{1}{2}\begin{pmatrix} m - Km^tK^{-1} & 0 \\ 0 & n - Kn^tK^{-1} \end{pmatrix}. \tag{1.27}$$

Analogously, for $M^{(2)}$ one obtains

$$M^{(2)} = \frac{1}{2}\begin{pmatrix} m + Km^tK^{-1} & 0 \\ 0 & n + Kn^tK^{-1} \end{pmatrix}. \tag{1.28}$$

At this point it is advantageous to make use of the explicit bases (1.10) for $\mathfrak{su}(2,2) \oplus \mathfrak{su}(4)$ introduced in the previous section. According to the discussion there, $\frac{1}{4}[\gamma^i, \gamma^j]$ with $i, j = 1, \ldots, 5$ generate the subalgebra $\mathfrak{so}(5) \subset \mathfrak{su}(4)$, while the commutators $\frac{1}{4}[\gamma^i, \gamma^j]$ and $\frac{\mathrm{i}}{4}[\gamma^i, \gamma^5]$ with $i, j = 1, \ldots, 4$ generate $\mathfrak{so}(4,1) \subset \mathfrak{su}(2,2)$. Further, the matrix K was chosen such that the following relations are satisfied

$$\gamma^i = K(\gamma^i)^tK^{-1}, \qquad [\gamma^i, \gamma^j] = -K[\gamma^i, \gamma^j]^tK^{-1}, \qquad i, j = 1, \ldots, 5. \tag{1.29}$$

These formulae reveal that the space $\mathscr{G}^{(0)}$ in the $\mathbb{Z}_4$-graded decomposition of $\mathfrak{psu}(2,2|4)$ coincides with the subalgebra $\mathfrak{so}(4,1) \oplus \mathfrak{so}(5) \subset \mathfrak{su}(2,2) \oplus \mathfrak{su}(4)$.

Similarly, comparing the structure of $M^{(2)}$ with equations (1.29), one finds that the space $\mathscr{G}^{(2)}$ is spanned by the matrices $\{\gamma^{1,2,3,4}, \mathrm{i}\gamma^5\} \in \mathfrak{su}(2,2)$ and $\{\mathrm{i}\gamma^i\} \in \mathfrak{su}(4)$, where $i = 1, \ldots, 5$. As we will see in section 1.4, these are the Lie algebra generators along the directions corresponding to the coset space $\mathrm{SU}(2,2) \times \mathrm{SU}(4)/\mathrm{SO}(4,1) \times \mathrm{SO}(5) = \mathrm{AdS}_5 \times \mathrm{S}^5$. The central element $\mathrm{i}\mathbb{1} \in \mathfrak{su}(2,2|4)$ also occurs in the projection $M^{(2)}$.

To complete the discussion of the $\mathbb{Z}_4$-graded decomposition, we also give the explicit formulae for the odd projections

$$
\begin{aligned}
M^{(1)} &= \frac{1}{2}\begin{pmatrix} 0 & \theta - \mathrm{i}K\eta^t K^{-1} \\ \eta + \mathrm{i}K\theta^t K^{-1} & 0 \end{pmatrix}, \\
M^{(3)} &= \frac{1}{2}\begin{pmatrix} 0 & \theta + \mathrm{i}K\eta^t K^{-1} \\ \eta - \mathrm{i}K\theta^t K^{-1} & 0 \end{pmatrix}.
\end{aligned}
\tag{1.30}
$$

1.2. Green–Schwarz string as coset model

For our further discussion, it is convenient to introduce an effective dimensionless string tension g, which for strings in $\mathrm{AdS}_5 \times \mathrm{S}^5$ is expressed through the radius R of S^5 and string slope α' as $g = R^2/2\pi\alpha'$. In the AdS/CFT correspondence this tension is related to the 't Hooft coupling constant λ as

$$
g = \frac{\sqrt{\lambda}}{2\pi}. \tag{1.31}
$$

We will consider a single closed string propagating in the $\mathrm{AdS}_5 \times \mathrm{S}^5$ space. Let coordinates σ and τ parametrize the string world-sheet which is a cylinder of circumference $2r$. For later convenience we assume the range of the world-sheet spatial coordinate σ to be $-r \leqslant \sigma \leqslant r$, where r is an arbitrary constant. The standard choice for a closed string is $r = \pi$. The string action is then

$$
S = \int \mathrm{d}\tau\, \mathrm{d}\sigma \mathscr{L}, \tag{1.32}
$$

where $\mathscr{L}$ is the Lagrangian density and the integration range for σ is assumed from $-r$ to r. In this section we outline the construction of the string Lagrangian and also analyze its global and local symmetries.

1.2.1. Lagrangian. Let $\mathfrak{g}$ be an element of the supergroup $\mathrm{SU}(2,2|4)$. Introduce the following 1-form with values in $\mathfrak{su}(2,2|4)$

$$
A = -\mathfrak{g}^{-1}\,\mathrm{d}\mathfrak{g} = A^{(0)} + A^{(2)} + A^{(1)} + A^{(3)}. \tag{1.33}
$$

Here on the right-hand side of the last formula we exhibited the $\mathbb{Z}_4$-decomposition of A, cf. equation (1.23). By construction, A has vanishing curvature $F = \mathrm{d}A - A \wedge A = 0$ or, in components,

$$
\partial_\alpha A_\beta - \partial_\beta A_\alpha - [A_\alpha, A_\beta] = 0. \tag{1.34}
$$

Now we postulate the following Lagrangian density describing a superstring in the $\mathrm{AdS}_5 \times \mathrm{S}^5$ background

$$
\mathscr{L} = -\frac{g}{2}\left[\gamma^{\alpha\beta}\,\mathrm{str}\left(A^{(2)}_\alpha A^{(2)}_\beta\right) + \kappa\epsilon^{\alpha\beta}\,\mathrm{str}\left(A^{(1)}_\alpha A^{(3)}_\beta\right)\right], \tag{1.35}
$$

which is the sum of the kinetic and the Wess–Zumino term. Here we use the convention $\epsilon^{\tau\sigma} = 1$ and $\gamma^{\alpha\beta} = h^{\alpha\beta}\sqrt{-h}$ is the Weyl-invariant combination[8] of the world-sheet metric $h_{\alpha\beta}$ with $\det\gamma = -1$. In the conformal gauge $\gamma^{\alpha\beta} = \mathrm{diag}(-1, 1)$. The parameter κ in front of the Wess–Zumino term has to be a real number to guarantee that the Lagrangian is a real (even) Grassmann element[9]. Indeed, assuming $\kappa = \kappa^*$ and taking into account the conjugation rule for the fermionic entries: $(\theta_1\theta_2)^* = \theta_2^*\theta_1^*$, as well as the cyclic property of the supertrace, we see that

$$\mathscr{L}^* = -\frac{g}{2}\left[\gamma^{\alpha\beta}\,\mathrm{str}\left(A_\alpha^{(2)\dagger}A_\beta^{(2)\dagger}\right) + \kappa\epsilon^{\alpha\beta}\,\mathrm{str}\left(A_\beta^{(3)\dagger}A_\alpha^{(1)\dagger}\right)\right] = \mathscr{L},$$

because all the projections $A^{(i)}$ are pseudo-Hermitian matrices obeying (1.3). Thus, the Lagrangian (1.35) is real.

Before we motivate formula (1.35), we would like to comment on the Wess–Zumino term. Originally, this term can be thought of as entering the action in the usual non-local fashion, i.e. as the following SO(4, 1) × SO(5)-invariant closed 3-form

$$\Theta_3 = \mathrm{str}(A^{(2)} \wedge A^{(3)} \wedge A^{(3)} - A^{(2)} \wedge A^{(1)} \wedge A^{(1)}) \tag{1.36}$$

integrated over a three-cycle with the boundary being a two-dimensional string world-sheet. The fact that Θ_3 is closed can be easily derived from the flatness condition for A. However, since the third cohomology group of the superconformal group is trivial the form Θ_3 appears to be exact

$$2\Theta_3 = \mathrm{d}\,\mathrm{str}(A^{(1)} \wedge A^{(3)}) \tag{1.37}$$

and, as a consequence, the Wess–Zumino term can be reduced to the two-dimensional integral, cf. equation (1.35).

Consider a transformation

$$\mathfrak{g} \to \mathfrak{g}\mathfrak{h}, \tag{1.38}$$

where $\mathfrak{h}$ belongs to SO(4, 1) × SO(5). Under this transformation the 1-form transforms as

$$A \to \mathfrak{h}^{-1}A\mathfrak{h} - \mathfrak{h}^{-1}\,\mathrm{d}\mathfrak{h}. \tag{1.39}$$

It is easy to see that for the $\mathbb{Z}_4$-components of A this transformation implies

$$A^{(1,2,3)} \to \mathfrak{h}^{-1}A^{(1,2,3)}\mathfrak{h}, \qquad A^{(0)} \to \mathfrak{h}^{-1}A^{(0)}\mathfrak{h} - \mathfrak{h}^{-1}\,\mathrm{d}\mathfrak{h}. \tag{1.40}$$

Thus, the component $A^{(0)}$ undergoes a gauge transformation, while all the other homogeneous components transform by the adjoint action.

By construction, the Lagrangian (1.35) depends on the group element $\mathfrak{g}$. However, as was shown above, under the right multiplication of $\mathfrak{g}$ with a local, i.e. σ- and τ-dependent element $\mathfrak{h} \in \mathrm{SO}(4,1) \times \mathrm{SO}(5)$, the homogeneous components $A^{(1)}$, $A^{(2)}$ and $A^{(3)}$ undergo a similarity transformation leaving the Lagrangian (1.35) invariant. Thus, the Lagrangian actually depends on a coset element from $\mathrm{SU}(2,2|4)/\mathrm{SO}(4,1) \times \mathrm{SO}(5)$, rather than on $\mathfrak{g} \in \mathrm{SU}(2,2|4)$.

Recall that in the $\mathbb{Z}_4$-decomposition of $A \in \mathfrak{su}(2,2|4)$ the central element $\mathrm{i}\mathbb{1}$ occurs in the projection $A^{(2)}$. As a result, under the right multiplication of $\mathfrak{g}$ with a group element from U(1) corresponding to $\mathrm{i}\mathbb{1}$, the component $A^{(2)}$ undergoes a shift

$$A^{(2)} \to A^{(2)} + c\cdot\mathrm{i}\mathbb{1}.$$

[8] Note the following formula for the inverse metric:

$$\gamma^{\alpha\beta} = \begin{pmatrix} -\gamma^{22} & \gamma^{12} \\ \gamma^{21} & -\gamma^{11} \end{pmatrix}.$$

[9] As we will see shortly, the requirement of κ-symmetry leaves two possibilities $\kappa = \pm 1$.

Since the supertrace of both the identity matrix and $A^{(2)}$ vanishes, this transformation leaves the Lagrangian (1.35) invariant. Thus, in addition to $\mathfrak{so}(4,1)\times\mathfrak{so}(5)$, we have an extra local $\mathfrak{u}(1)$-symmetry induced by the central element $i\mathbb{1}$. Clearly, this symmetry can be used to gauge away the trace part of $A^{(2)}$. Thus, in what follows we will assume that $A^{(2)}$ is chosen to be traceless, which can be viewed as the gauge fixing condition for these $\mathfrak{u}(1)$-transformations.

The group of global symmetry transformations of the Lagrangian (1.35) coincides with PSU(2, 2|4). Indeed, PSU(2, 2|4) acts on the coset space (1.1) by multiplication from the left. If $\mathfrak{g}\in \mathrm{PSU}(2,2|4)$ is a coset space representative and G is an arbitrary group element from PSU(2, 2|4), then the action of G on $\mathfrak{g}$ is as follows:

$$G:\mathfrak{g}\to\mathfrak{g}', \tag{1.41}$$

where $\mathfrak{g}'$ is determined from the following equation:

$$G\cdot\mathfrak{g}=\mathfrak{g}'\mathfrak{h}. \tag{1.42}$$

Here $\mathfrak{g}'$ is a new coset representative and $\mathfrak{h}$ is a 'compensating' local element from $\mathrm{SO}(4,1)\times\mathrm{SO}(5)$. Because of the local invariance under $\mathrm{SO}(4,1)\times\mathrm{SO}(5)$ the Lagrangian (1.35) is also invariant under global PSU(2, 2|4)-transformations. The detailed discussion of these global symmetry transformations will be postponed till section 1.4.

Further justification of the Lagrangian (1.35) comes from the fact that when restricted to bosonic variables only, it reproduces the usual Polyakov action for bosonic strings propagating in the $\mathrm{AdS}_5\times\mathrm{S}^5$ geometry. We will present the corresponding derivation in section 1.5.2.

Our next goal is to derive the equations of motion following from equation (1.35). We first note that if M_1 and M_2 are two supermatrices then

$$\mathrm{str}(\Omega^k(M_1)M_2)=\mathrm{str}(M_1\Omega^{4-k}(M_2)) \tag{1.43}$$

for $k=1,2,3$. By using this property, the variation of the Lagrangian density can be cast in the form

$$\delta\mathscr{L}=-\mathrm{str}(\delta A_\alpha\Lambda^\alpha), \tag{1.44}$$

where

$$\Lambda^\alpha=g\big[\gamma^{\alpha\beta}A^{(2)}_\beta-\tfrac{1}{2}\kappa\epsilon^{\alpha\beta}\big(A^{(1)}_\beta-A^{(3)}_\beta\big)\big]. \tag{1.45}$$

Taking into account that the variation of A_α is

$$\delta A_\alpha=-\delta(\mathfrak{g}^{-1}\partial_\alpha\mathfrak{g})=-\mathfrak{g}^{-1}\delta\mathfrak{g}A_\alpha-\mathfrak{g}^{-1}\partial_\alpha(\delta\mathfrak{g}),$$

we obtain

$$\delta\mathscr{L}=\mathrm{str}[\mathfrak{g}^{-1}\delta\mathfrak{g}A_\alpha\Lambda^\alpha+\mathfrak{g}^{-1}\partial_\alpha(\delta\mathfrak{g})\Lambda^\alpha].$$

Finally, integrating the last term by parts and omitting the total derivative contribution, we arrive at the following expression for the variation of the Lagrangian density:

$$\delta\mathscr{L}=-\mathrm{str}[\mathfrak{g}^{-1}\delta\mathfrak{g}(\partial_\alpha\Lambda^\alpha-[A_\alpha,\Lambda^\alpha])]. \tag{1.46}$$

Thus, if we regard $\partial_\alpha\Lambda^\alpha-[A_\alpha,\Lambda^\alpha]$ as an element of $\mathfrak{su}(2,2|4)$, then the equations of motion read as

$$\partial_\alpha\Lambda^\alpha-[A_\alpha,\Lambda^\alpha]=\varrho\cdot\mathbb{1}, \tag{1.47}$$

where the coefficient ϱ is found by taking the trace of both sides of the last equation. Since $\mathfrak{psu}(2,2|4)$ is understood as the quotient of $\mathfrak{su}(2,2|4)$ over its one-dimensional center, in $\mathfrak{psu}(2,2|4)$ the equations of motion take the form

$$\partial_\alpha\Lambda^\alpha-[A_\alpha,\Lambda^\alpha]=0. \tag{1.48}$$

The single equation (1.48) can be projected on various $\mathbb{Z}_4$-components. First, one finds that the projection on $\mathscr{G}^{(0)}$ vanishes. Second, for the projection on $\mathscr{G}^{(2)}$ we get

$$\partial_\alpha\big(\gamma^{\alpha\beta}A^{(2)}_\beta\big) - \gamma^{\alpha\beta}\big[A^{(0)}_\alpha, A^{(2)}_\beta\big] + \tfrac{1}{2}\kappa\epsilon^{\alpha\beta}\big(\big[A^{(1)}_\alpha, A^{(1)}_\beta\big] - \big[A^{(3)}_\alpha, A^{(3)}_\beta\big]\big) = 0, \tag{1.49}$$

while the for projections on $\mathscr{G}^{(1,3)}$ one finds

$$\begin{aligned} &\gamma^{\alpha\beta}\big[A^{(3)}_\alpha, A^{(2)}_\beta\big] + \kappa\epsilon^{\alpha\beta}\big[A^{(2)}_\alpha, A^{(3)}_\beta\big] = 0,\\ &\gamma^{\alpha\beta}\big[A^{(1)}_\alpha, A^{(2)}_\beta\big] - \kappa\epsilon^{\alpha\beta}\big[A^{(2)}_\alpha, A^{(1)}_\beta\big] = 0. \end{aligned} \tag{1.50}$$

In deriving these equations we also used the condition of zero curvature for the connection A_α. Introducing the tensors

$$\mathrm{P}^{\alpha\beta}_\pm = \tfrac{1}{2}(\gamma^{\alpha\beta} \pm \kappa\epsilon^{\alpha\beta}), \tag{1.51}$$

equations (1.50) can be written as

$$\mathrm{P}^{\alpha\beta}_-\big[A^{(2)}_\alpha, A^{(3)}_\beta\big] = 0, \qquad \mathrm{P}^{\alpha\beta}_+\big[A^{(2)}_\alpha, A^{(1)}_\beta\big] = 0. \tag{1.52}$$

We further note that for $\kappa = \pm 1$ the tensors $\mathrm{P}_\pm$ are orthogonal projectors:

$$\mathrm{P}^{\alpha\beta}_+ + \mathrm{P}^{\alpha\beta}_- = \gamma^{\alpha\beta}, \qquad \mathrm{P}^{\alpha\delta}_\pm \mathrm{P}^{\beta}_{\pm\delta} = \mathrm{P}^{\alpha\beta}_\pm, \qquad \mathrm{P}^{\alpha\delta}_\pm \mathrm{P}^{\beta}_{\mp\delta} = 0. \tag{1.53}$$

Further, we emphasize the relation between the equations of motion and the global symmetries of the model (the Noether theorem). Consider the following current:

$$J^\alpha = \mathfrak{g}\Lambda^\alpha\mathfrak{g}^{-1}. \tag{1.54}$$

Due to equation (1.48), this current is conserved:

$$\partial_\alpha J^\alpha = 0. \tag{1.55}$$

In fact, J^α is nothing else but the Noether current corresponding to global PSU(2,2|4)-symmetry transformations. The corresponding conserved charge Q is given by the following integral of the J^τ component:

$$\mathrm{Q} = \int_{-r}^{r} \mathrm{d}\sigma\, J^\tau = g\int_{-r}^{r} \mathrm{d}\sigma\, \mathfrak{g}\left[\gamma^{\tau\tau}A^{(2)}_\tau + \gamma^{\tau\sigma}A^{(2)}_\sigma - \frac{\kappa}{2}\big(A^{(1)}_\sigma - A^{(3)}_\sigma\big)\right]\mathfrak{g}^{-1}. \tag{1.56}$$

It is worth pointing out that in the matrix representation the current J^α is an element of $\mathfrak{su}(2,2|4)$ and, for this reason, only its traceless part is conserved.

Finally, we also have equations of motion for the world-sheet metric which are equivalent to vanishing the world-sheet stress-tensor

$$\mathrm{str}\big(A^{(2)}_\alpha A^{(2)}_\beta\big) - \tfrac{1}{2}\gamma_{\alpha\beta}\gamma^{\rho\delta}\,\mathrm{str}\big(A^{(2)}_\rho A^{(2)}_\delta\big) = 0. \tag{1.57}$$

These equations are known as the Virasoro constraints and they reflect the two-dimensional reparametrization invariance of the string action.

In summary, we presented a construction of the superstring Lagrangian based on the flat connection A. The Lagrangian comprises degrees of freedom corresponding to the coset space (1.1) and it is invariant with respect to the global PSU(2, 2|4)-symmetry transformations. The flat connection A allows one to introduce a new current J^α which is conserved due to the superstring equations of motion; the corresponding conserved charge is a generator of these global symmetry transformations.

1.2.2. Parity transform and time reversal. In section (1.1.2) we introduced a continuous group $\{\delta_\rho\}$ of automorphisms of $\mathfrak{sl}(4|4)$. For ρ restricted to the unit circle this group also becomes an automorphism group of $\mathfrak{psu}(2,2|4)$. In particular, the automorphisms δ_{-1} and $\delta_{\pm i}$ are inner. Here we will argue that the action of the elements $\delta_{\pm i}$ on the string Lagrangian (1.35) can be essentially viewed as the parity transformation or, equivalently, as the time reversal operation.

More generally, we start our analysis by considering the following transformation:

$$\mathfrak{g}' = U\mathfrak{g}U^{-1}, \tag{1.58}$$

where $\mathfrak{g} \in \mathrm{PSU}(2,2|4)$ and U is some global (constant) bosonic matrix. The matrix U should not however belong to $\mathrm{SO}(4,1)\times\mathrm{SO}(5)$, as in the opposite case we have already established the invariance of the string Lagrangian: it is separately invariant under multiplication of $\mathfrak{g}$ by a global element U from the left and by a local element V from the right. Under transformation (1.58) the connection $A = -\mathfrak{g}^{-1}\,\mathrm{d}\mathfrak{g}$ undergoes a change

$$A \to A' = UAU^{-1}.$$

Imposing an extra requirement that U commutes with $\mathcal{K}$, we obtain

$$\Omega(A') = -\mathcal{K}(UAU^{-1})^{\mathrm{st}}\mathcal{K}^{-1} = (U^t)^{-1}\Omega(A)U^t. \tag{1.59}$$

This formula allows us to construct the $\mathbb{Z}_4$-graded decomposition of the transformed connection A'. First, we look at the projection $A'^{(2)}$

$$A'^{(2)} = \tfrac{1}{4}[A' - \Omega(A') + \Omega^2(A') - \Omega^3(A')], \tag{1.60}$$

which, upon the usage of equation (1.59), takes the form

$$A'^{(2)} = \tfrac{1}{4}[U(A + \Omega^2(A))U^{-1} - (U^t)^{-1}(\Omega(A) + \Omega^3(A))U^t]. \tag{1.61}$$

Substituting here the $\mathbb{Z}_4$-graded decomposition (1.33) of A, we see that

$$A'^{(2)} = \tfrac{1}{2}[U(A^{(0)} + A^{(2)})U^{-1} - (U^t)^{-1}(A^{(0)} - A^{(2)})U^t]. \tag{1.62}$$

Analogous considerations allow one to establish the formulae for the odd components of the transformed connection

$$\begin{aligned} A'^{(1)} &= \tfrac{1}{2}[U(A^{(1)} + A^{(3)})U^{-1} + (U^t)^{-1}(A^{(1)} - A^{(3)}))U^t],\\ A'^{(3)} &= \tfrac{1}{2}[U(A^{(1)} + A^{(3)})U^{-1} - (U^t)^{-1}(A^{(1)} - A^{(3)}))U^t]. \end{aligned} \tag{1.63}$$

These expressions suggest to consider the following two cases. The first one corresponds to taking U such that

$$U^tU = \mathbb{1}, \qquad [U,\mathcal{K}] = 0. \tag{1.64}$$

With this choice the transformation formulae (1.62) and (1.63) simplify to

$$A'^{(2)} = UA^{(2)}U^{-1}, \qquad A'^{(1)} = UA^{(1)}U^{-1}, \qquad A'^{(3)} = UA^{(3)}U^{-1}. \tag{1.65}$$

We thus see that the Lagrangian (1.35) remains invariant[10], however, there is nothing new here because the group singled out by the requirements (1.64) is just a subgroup of $\mathrm{SO}(4,1)\times\mathrm{SO}(5)$.

The second case corresponds to imposing the following requirements:

$$U^tU = \Upsilon, \qquad [U,\mathcal{K}] = 0, \tag{1.66}$$

[10] In fact, the Lagrangian remains invariant under a milder assumption on U, namely, $U^tU = \mathrm{e}^{\mathrm{i}\alpha}\mathbb{1}$, where $\mathrm{e}^{\mathrm{i}\alpha}$ is an arbitrary phase. However, this phase plays no role – being absorbed into U, it drops out of the similarity transformation (1.58). The matrix U corresponding to δ_{-1} is $U = \mathrm{i}\Upsilon$, so that it commutes with $\mathcal{K}$ and obeys $U^tU = -\mathbb{1}$. Thus, the action of δ_{-1} leaves the Lagrangian invariant.

where we omitted an unessential overall phase in front of Υ, see the footnote 5. Since for any odd matrix M one has $\Upsilon M \Upsilon^{-1} = -M$, expressions (1.62) and (1.63) reduce to

$$A'^{(2)} = UA^{(2)}U^{-1}, \qquad A'^{(1)} = UA^{(3)}U^{-1}, \qquad A'^{(3)} = UA^{(1)}U^{-1}.$$

Thus, in essence, the transformation above exchanges the projections $A^{(1)}$ and $A^{(3)}$. For this reason, it does not leave the Lagrangian (1.35) invariant, rather it changes the sign in front of the Wess–Zumino term.

As the simplest solutions to equations (1.66), we can take

$$U = \begin{pmatrix} \mathrm{i}^{\frac{1}{2}}\mathbb{1}_4 & 0 \\ 0 & \mathrm{i}^{-\frac{1}{2}}\mathbb{1}_4 \end{pmatrix} = \mathrm{e}^{\mathrm{i}\frac{\pi}{4}\Upsilon}, \tag{1.67}$$

which corresponds to the action of δ_i. We identify U as a matrix corresponding to the parity transformation $\mathscr{P} \equiv U$. Indeed, under the map $\sigma \to -\sigma$ the Wess–Zumino term changes its sign[11]. This sign change can be then compensated by transformation (1.58) with U given by (1.67). Thus, under the combined transformation

$$\sigma \to -\sigma, \qquad \mathfrak{g} \to \mathscr{P}\mathfrak{g}\mathscr{P}^{-1} \tag{1.68}$$

the action remains invariant. Under $\mathscr{P}$ a supermatrix M transforms as follows

$$M = \begin{pmatrix} m & \theta \\ \eta & n \end{pmatrix} \to \mathscr{P}M\mathscr{P}^{-1} = \begin{pmatrix} m & \mathrm{i}\theta \\ -\mathrm{i}\eta & n \end{pmatrix}, \tag{1.69}$$

i.e. fermions are multiplied by $\pm i$ which can be identified as their intrinsic parity.

Before the gauge fixing, σ and τ variables enter the sigma model action on equal footing. Therefore, one can equally regard the action of U together with the change $\tau \to -\tau$ as the time reversal operation. In the gauge-fixed theory the time reversal operation acts differently. We will return to this issue in section 3.

1.2.3. Kappa-symmetry. Kappa-symmetry is a local fermionic symmetry of the Green–Schwarz superstring. It generalizes the local fermionic symmetries first discovered for massive and massless superparticles and its presence is crucial to ensure the spacetime supersymmetry of the physical spectrum. Here we establish κ-symmetry transformations associated with the Lagrangian (1.35).

Deriving κ-symmetry. Recall that the action of the global symmetry group PSU(2, 2|4) is realized on a coset element by multiplication from the left. In this respect, κ-symmetry transformations can be viewed as the *right local* action of $G = \exp\epsilon$ on the coset representative $\mathfrak{g}$:

$$\mathfrak{g} \cdot G = \mathfrak{g}'\mathfrak{h}, \tag{1.70}$$

where $\epsilon \equiv \epsilon(\tau, \sigma)$ is a local fermionic parameter taking values in $\mathfrak{psu}(2, 2|4)$. Here $\mathfrak{h}$ is a compensating element from SO(4, 1) × SO(5). The main difference with the case of global symmetry is that for arbitrary ϵ the action is not invariant under transformation (1.70). Below we find the conditions on ϵ which guarantee the invariance of the action.

First, we note that under local multiplication from the right the 1-form A transforms as follows

$$\delta_\epsilon A = -\mathrm{d}\epsilon + [A, \epsilon]. \tag{1.71}$$

[11] The pseudo-tensor $\epsilon^{\alpha\beta}$ does not change its sign under $\sigma \to \sigma$ or $\tau \to -\tau$.

The $\mathbb{Z}_4$-decomposition of this equation gives

$$\begin{aligned}
\delta_\epsilon A^{(1)} &= -\mathrm{d}\epsilon^{(1)} + [A^{(0)}, \epsilon^{(1)}] + [A^{(2)}, \epsilon^{(3)}],\\
\delta_\epsilon A^{(3)} &= -\mathrm{d}\epsilon^{(3)} + [A^{(2)}, \epsilon^{(1)}] + [A^{(0)}, \epsilon^{(3)}],\\
\delta_\epsilon A^{(2)} &= [A^{(1)}, \epsilon^{(1)}] + [A^{(3)}, \epsilon^{(3)}],\\
\delta_\epsilon A^{(0)} &= [A^{(3)}, \epsilon^{(1)}] + [A^{(1)}, \epsilon^{(3)}],
\end{aligned} \tag{1.72}$$

where we have assumed that $\epsilon = \epsilon^{(1)} + \epsilon^{(3)}$. By using these formulae, we find for the variation of the Lagrangian density

$$\begin{aligned}
-\frac{2}{g}\delta_\epsilon \mathscr{L} = {}& \delta\gamma^{\alpha\beta}\,\mathrm{str}\big(A^{(2)}_\alpha A^{(2)}_\beta\big) - 2\gamma^{\alpha\beta}\,\mathrm{str}\big(\big[A^{(1)}_\alpha, A^{(2)}_\beta\big]\epsilon^{(1)} + \big[A^{(3)}_\alpha, A^{(2)}_\beta\big]\epsilon^{(3)}\big)\\
&+ \kappa\epsilon^{\alpha\beta}\,\mathrm{str}\big(\partial_\alpha A^{(3)}_\beta \epsilon^{(1)} - \partial_\alpha A^{(1)}_\beta \epsilon^{(3)} + \big[A^{(0)}_\alpha, \epsilon^{(1)}\big]A^{(3)}_\beta + \big[A^{(2)}_\alpha, \epsilon^{(3)}\big]A^{(3)}_\beta\\
&+ A^{(1)}_\alpha\big[A^{(0)}_\beta, \epsilon^{(3)}\big] + A^{(1)}_\alpha\big[A^{(2)}_\beta, \epsilon^{(1)}\big]\big).
\end{aligned} \tag{1.73}$$

Note that the derivatives of ϵ have been eliminated by integrating by parts and subsequently neglecting the corresponding total derivatives. The variation of the world-sheet metric is left unspecified. The flatness condition (1.34) implies

$$\begin{aligned}
\epsilon^{\alpha\beta}\partial_\alpha A^{(1)}_\beta &= \epsilon^{\alpha\beta}\big[A^{(0)}_\alpha, A^{(1)}_\beta\big] + \epsilon^{\alpha\beta}\big[A^{(2)}_\alpha, A^{(3)}_\beta\big],\\
\epsilon^{\alpha\beta}\partial_\alpha A^{(3)}_\beta &= \epsilon^{\alpha\beta}\big[A^{(0)}_\alpha, A^{(3)}_\beta\big] + \epsilon^{\alpha\beta}\big[A^{(1)}_\alpha, A^{(2)}_\beta\big].
\end{aligned} \tag{1.74}$$

Taking into account these formulae, we obtain

$$-\frac{2}{g}\delta_\epsilon \mathscr{L} = \delta\gamma^{\alpha\beta}\,\mathrm{str}\big(A^{(2)}_\alpha A^{(2)}_\beta\big) - 4\mathrm{str}\big(\mathrm{P}^{\alpha\beta}_+\big[A^{(1)}_\beta, A^{(2)}_\alpha\big]\epsilon^{(1)} + \mathrm{P}^{\alpha\beta}_-\big[A^{(3)}_\beta, A^{(2)}_\alpha\big]\epsilon^{(3)}\big).$$

For any vector V^α we introduce the projections $V^\alpha_\pm$

$$V^\alpha_\pm = \mathrm{P}^{\alpha\beta}_\pm V_\beta$$

so that the variation of the Lagrangian acquires the form

$$-\frac{2}{g}\delta_\epsilon \mathscr{L} = \delta\gamma^{\alpha\beta}\,\mathrm{str}\big(A^{(2)}_\alpha A^{(2)}_\beta\big) - 4\mathrm{str}\big(\big[A^{(1),\alpha}_+, A^{(2)}_{\alpha,-}\big]\epsilon^{(1)} + \big[A^{(3),\alpha}_-, A^{(2)}_{\alpha,+}\big]\epsilon^{(3)}\big). \tag{1.75}$$

We further note that from the condition $\mathrm{P}^{\alpha\beta}_\pm A_{\beta,\mp} = 0$ the components $A_{\tau,\pm}$ and $A_{\sigma,\pm}$ are proportional

$$A_{\tau,\pm} = -\frac{\gamma^{\tau\sigma} \mp \kappa}{\gamma^{\tau\tau}} A_{\sigma,\pm}. \tag{1.76}$$

The crucial point of our construction is the following ansatz for the κ-symmetry parameters $\epsilon^{(1)}$ and $\epsilon^{(3)}$:

$$\epsilon^{(1)} = A^{(2)}_{\alpha,-}\kappa^{(1),\alpha}_+ + \kappa^{(1),\alpha}_+ A^{(2)}_{\alpha,-}, \qquad \epsilon^{(3)} = A^{(2)}_{\alpha,+}\kappa^{(3),\alpha}_- + \kappa^{(3),\alpha}_- A^{(2)}_{\alpha,+}. \tag{1.77}$$

Here $\kappa^{(i),\alpha}_\pm$ are new independent parameters of κ-symmetry transformation which are homogeneous elements of degree $i = 1, 3$ with respect to Ω. The correct degree of ϵ is inherited from the properties of Ω (see footnote 2). For instance, one has

$$\begin{aligned}
&\Omega\big(A^{(2)}_{\alpha,-}\kappa^{(1),\alpha}_+ + \kappa^{(1),\alpha}_+ A^{(2)}_{\alpha,-}\big)\\
&\quad = -\Omega\big(\kappa^{(1),\alpha}_+\big)\Omega\big(A^{(2)}_{\alpha,-}\big) - \Omega\big(A^{(2)}_{\alpha,-}\big)\Omega\big(\kappa^{(1),\alpha}_+\big) = i\big(A^{(2)}_{\alpha,-}\kappa^{(1),\alpha}_+ + \kappa^{(1),\alpha}_+ A^{(2)}_{\alpha,-}\big).
\end{aligned}$$

Finally, $\epsilon^{(1,3)} \in \mathfrak{su}(2,2|4)$ provided the matrices $\kappa^{(1)}$ and $\kappa^{(3)}$ satisfy the following reality conditions

$$H\kappa^{(1)} - (\kappa^{(1)})^\dagger H = 0, \qquad H\kappa^{(3)} - (\kappa^{(3)})^\dagger H = 0. \tag{1.78}$$

As was explained in section 1.1.2, the (traceless) component $A^{(2)}$ taking values in $\mathfrak{su}(2,2|4)$ can be expanded as

$$A^{(2)} = \begin{pmatrix} m^i\gamma^i & 0 \\ 0 & n^i\gamma^i \end{pmatrix}, \tag{1.79}$$

where γ^i are the SO(5) Dirac matrices. The coefficients n^i are all imaginary, while m^i are real for $i = 1, \ldots, 4$ and imaginary for $i = 5$. Thus, assuming $A^{(2)}$ to be traceless we can further write

$$A^{(2)}_{\alpha,\pm}A^{(2)}_{\beta,\pm} = \begin{pmatrix} m^i_{\alpha,\pm}m^j_{\beta,\pm}\gamma^i\gamma^j & 0 \\ 0 & n^i_{\alpha,\pm}n^j_{\beta,\pm}\gamma^i\gamma^j \end{pmatrix}.$$

Since the chiral components $A_{\tau,\pm}$ and $A_{\sigma,\pm}$ are proportional to each other, see equation (1.76), we can rewrite the last formula as

$$A^{(2)}_{\alpha,\pm}A^{(2)}_{\beta,\pm} = \begin{pmatrix} m^i_{\alpha,\pm}m^j_{\beta,\pm}\frac{1}{2}\{\gamma^i,\gamma^j\} & 0 \\ 0 & n^i_{\alpha,\pm}n^j_{\beta,\pm}\frac{1}{2}\{\gamma^i,\gamma^j\} \end{pmatrix}.$$

This expression can be concisely written as

$$A^{(2)}_{\alpha,\pm}A^{(2)}_{\beta,\pm} = \begin{pmatrix} m^i_{\alpha,\pm}m^i_{\beta,\pm} & 0 \\ 0 & n^i_{\alpha,\pm}n^i_{\beta,\pm} \end{pmatrix} = \frac{1}{8}\Upsilon\,\mathrm{str}\big(A^{(2)}_{\alpha,\pm}A^{(2)}_{\beta,\pm}\big) + c_{\alpha\beta}\,\mathbb{1}_8 \tag{1.80}$$

where $c_{\alpha\beta} = \frac{1}{2}\big(m^i_{\alpha,\pm}m^i_{\beta,\pm} + n^i_{\alpha,\pm}n^i_{\beta,\pm}\big)$ and Υ is the hypercharge (1.15). In other words, the product of two $A^{(2)}$'s entering the variation upon substitution of the ansatz (1.77) appears to be just a linear combination of two matrices, one of them being the identity matrix and the other being Υ.

Therefore, for the variation of the Lagrangian density we find

$$-\frac{2}{g}\delta_\epsilon\mathscr{L} = \delta\gamma^{\alpha\beta}\,\mathrm{str}\big(A^{(2)}_\alpha A^{(2)}_\beta\big) - \frac{1}{2}\,\mathrm{str}\big(A^{(2)}_{\alpha,-}A^{(2)}_{\beta,-}\big)\,\mathrm{str}\big(\Upsilon[\kappa^{(1),\beta}_+, A^{(1),\alpha}_+]\big)$$
$$-\frac{1}{2}\,\mathrm{str}\big(A^{(2)}_{\alpha,+}A^{(2)}_{\beta,+}\big)\,\mathrm{str}(\Upsilon[\kappa^{(3),\beta}_-, A^{(3),\alpha}_-]),$$

where the contribution of the term in equation (1.80) proportional to the identity matrix drops out. It is now clear that this variation vanishes provided we assume the following transformation rule for the world-sheet metric under κ-symmetry transformations:

$$\delta\gamma^{\alpha\beta} = \tfrac{1}{4}\,\mathrm{str}\big(\Upsilon\big(\big[\kappa^{(1),\alpha}_+, A^{(1),\beta}_+\big] + \big[\kappa^{(1),\beta}_+, A^{(1),\alpha}_+\big] + \big[\kappa^{(3),\alpha}_-, A^{(3),\beta}_-\big] + \big[\kappa^{(3),\beta}_-, A^{(3),\alpha}_-\big]\big)\big).$$

This variation is an even symmetric tensor satisfying the identity $\gamma_{\alpha\beta}\delta\gamma^{\alpha\beta} = 0$. Moreover, the reality conditions for A and κ guarantee that the variation $\delta\gamma^{\alpha\beta}$ is a tensor with real components.

It is useful to note that the projectors $\mathrm{P}^{\alpha\beta}_\pm$ satisfy the following important identity:

$$\mathrm{P}^{\alpha\gamma}_\pm\,\mathrm{P}^{\beta\delta}_\pm = \mathrm{P}^{\beta\gamma}_\pm\,\mathrm{P}^{\alpha\delta}_\pm. \tag{1.81}$$

This identity allows one to rewrite the variation of the metric in a simpler form

$$\delta\gamma^{\alpha\beta} = \tfrac{1}{2}\,\mathrm{tr}\big(\big[\kappa^{(1),\alpha}_+, A^{(1),\beta}_+\big] + \big[\kappa^{(3),\alpha}_-, A^{(3),\beta}_-\big]\big), \tag{1.82}$$

where we used the fact that the supertrace of any matrix with an insertion of Υ is the same as the usual trace of this matrix. It is worthwhile to point out that in our derivation of κ-symmetry transformations we exploited the fact that $\mathrm{P}^{\alpha\beta}_\pm$ are orthogonal projectors and, therefore, the realization of κ-symmetry requires the parameter κ in front of the Wess–Zumino term to take one of the values $\kappa = \pm 1$.

On-shell rank of κ-symmetry transformations. The next important question is to understand how many fermionic degrees of freedom can be gauged away on-shell by means of κ-symmetry. To this end, one can make use of the light-cone gauge[12]. Generically, the light-cone coordinates $x_\pm$ are introduced by making linear combinations of one field corresponding to the time direction from AdS_5 and one field from S^5. Without loss of generality we can assume that the transversal fluctuations are all suppressed and the corresponding element $A^{(2)}$ has the form

$$A^{(2)} = \begin{pmatrix} \mathrm{i}x\gamma^5 & 0 \\ 0 & \mathrm{i}y\gamma^5 \end{pmatrix}. \tag{1.83}$$

Indeed, the matrix $\mathrm{i}\gamma^5$ corresponds to the time direction in AdS_5 and any element from the tangent space to S^5 can be brought to γ^5 by means of an $\mathfrak{su}(4)$ transformation.

Consider first the κ-symmetry parameter $\epsilon^{(1)}$. In the present context, going on-shell means the fulfilment of the Virasoro constraint $\mathrm{str}\big(A^{(2)}_{\alpha,-}A^{(2)}_{\beta,-}\big) = 0$, the latter boils down to $x^2 = y^2$, i.e. to $y = \pm x$. According to equation (1.72), we have

$$\epsilon^{(1)} = A^{(2)}_{\tau,-}\varkappa + \varkappa A^{(2)}_{\tau,-}, \qquad \varkappa \equiv \kappa_+^{(1),\tau} - \frac{\gamma^{\tau\tau}}{\gamma^{\tau\sigma} \mp \kappa}\kappa_+^{(1),\sigma}. \tag{1.84}$$

Picking, e.g., the solution $y = x$, we compute the element $\epsilon^{(1)}$. Plugging equation (1.83) into equation (1.13) and assuming for the moment that $\varkappa$ is generic, i.e. that it depends on 32 independent (real) fermionic variables, we obtain

$$\epsilon^{(1)} = 2\mathrm{i}x \begin{pmatrix} 0 & \varepsilon \\ -\varepsilon^\dagger\Sigma & 0 \end{pmatrix}, \tag{1.85}$$

where ε is the following matrix:

$$\varepsilon = \begin{pmatrix} \varkappa_{11} & \varkappa_{12} & 0 & 0 \\ \varkappa_{21} & \varkappa_{22} & 0 & 0 \\ 0 & 0 & -\varkappa_{33} & -\varkappa_{34} \\ 0 & 0 & -\varkappa_{43} & -\varkappa_{44} \end{pmatrix} \tag{1.86}$$

and $\varkappa_{ij}$ are the entries of the matrix $\varkappa$. As we see, the matrix ε depends on eight independent complex fermionic parameters. Now we can account for the fact that the odd matrix $\varkappa$ belongs to the homogeneous component $\mathscr{G}^{(1)}$ which reduces the number of independent fermions by half. As a result, $\epsilon^{(1)}$ depends on eight real fermionic parameters. A similar analysis shows that $\epsilon^{(3)}$ will also depend on other eight real fermions. Thus, in total $\epsilon^{(1)}$ and $\epsilon^{(3)}$ involve 16 real fermions and these are those degrees of freedom which can be gauged away by κ-symmetry. The gauge-fixed coset model will therefore involve 16 physical fermions only.

The above analysis, especially equations (1.85) and (1.86), show that κ-symmetry suffice to bring a generic odd element of $\mathfrak{su}(2,2|4)$ to the following form:

$$\left(\begin{array}{cccc|cccc} 0 & 0 & 0 & 0 & 0 & 0 & \bullet & \bullet \\ 0 & 0 & 0 & 0 & 0 & 0 & \bullet & \bullet \\ 0 & 0 & 0 & 0 & \bullet & \bullet & 0 & 0 \\ 0 & 0 & 0 & 0 & \bullet & \bullet & 0 & 0 \\ \hline 0 & 0 & \bullet & \bullet & 0 & 0 & 0 & 0 \\ 0 & 0 & \bullet & \bullet & 0 & 0 & 0 & 0 \\ \bullet & \bullet & 0 & 0 & 0 & 0 & 0 & 0 \\ \bullet & \bullet & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right), \tag{1.87}$$

[12] String theory in the light-cone gauge will be treated in great detail in the following section.

where bullets stand for odd elements which cannot be gauged away by κ-symmetry transformations. We thus consider (1.87) as a convenient κ-symmetry gauge choice and we will implement it in our construction of the light-cone string action in the following section.

1.3. Integrability of classical superstrings

In this section we show that the nonlinear sigma model describing strings on $AdS_5 \times S^5$ is a classical two-dimensional integrable system. This opens up the possibility of analyzing it by means of techniques developed in the theory of integrable models. We start with recalling the general concept of integrability and then we demonstrate integrability of the string sigma model by constructing the zero-curvature representation of the corresponding equations of motion. Finally, we discuss the relationship between integrability and the local, and global symmetries of the string model.

1.3.1. General concept of integrability. The classical inverse scattering method, i.e. the method of finding a certain class of solutions of nonlinear integrable partial differential equations, is based on a remarkable observation that a two-dimensional partial differential equation arises as the consistency condition of the following overdetermined system of equations:

$$\frac{\partial\Psi}{\partial\sigma} = L_\sigma(\sigma,\tau,z)\Psi, \qquad \frac{\partial\Psi}{\partial\tau} = L_\tau(\sigma,\tau,z)\Psi, \tag{1.88}$$

which is sometimes referred to as the fundamental linear problem. Here $\Psi \equiv \Psi(\sigma,\tau,z)$ is a vector of rank $\mathfrak{r}$ and $L_\sigma \equiv L_\sigma(\sigma,\tau,z)$ and $L_\tau \equiv L_\tau(\sigma,\tau,z)$ are properly chosen $\mathfrak{r}\times\mathfrak{r}$ matrices. Both Ψ and L_σ, L_τ depend on an additional *spectral parameter* z taking values in the complex plane[13]. Differentiating the first equation in (1.88) with respect to τ and the second one with respect to σ, we obtain

$$\frac{\partial^2\Psi}{\partial\tau\partial\sigma} = \partial_\tau L_\sigma\Psi + L_\sigma\partial_\tau\Psi = (\partial_\tau L_\sigma + L_\sigma L_\tau)\Psi,$$

$$\frac{\partial^2\Psi}{\partial\sigma\partial\tau} = \partial_\sigma L_\tau\Psi + L_\tau\partial_\sigma\Psi = (\partial_\sigma L_\tau + L_\tau L_\sigma)\Psi,$$

which implies the fulfilment of the following consistency condition:

$$\partial_\tau L_\sigma - \partial_\sigma L_\tau + [L_\sigma, L_\tau] = 0$$

for all values of the spectral parameter. If we introduce a two-dimensional non-Abelian connection L_α with components L_τ and L_σ, then the consistency condition derived above can be reinstated as vanishing of the curvature of L_α

$$\partial_\alpha L_\beta - \partial_\beta L_\alpha - [L_\alpha, L_\beta] = 0. \tag{1.89}$$

The matrices L_τ and L_σ must be chosen in such a way that the zero-curvature condition above should imply the fulfilment of the original differential equation for all values of the spectral parameter. A connection L_α with these properties is known as the Lax connection (or the Lax pair), while equation (1.89) as the zero-curvature (Lax) representation of an integrable partial differential equation.

[13] In more complicated situations the spectral parameter can live on a higher genus Riemann surface.

For a given integrable partial differential equation the Lax connection is by no means unique. Even the rank $\mathfrak{r}$ of the matrices L_α can be different for different Lax representations. Also, the condition of zero curvature (1.89) is invariant with respect to the gauge transformations

$$L_\alpha \to L'_\alpha = hL_\alpha h^{-1} + \partial_\alpha h h^{-1}, \tag{1.90}$$

where h is an arbitrary matrix, in general depending on dynamical variables of the model and the spectral parameter.

Conservation laws. The usefulness of the Lax connection lies in the fact that for a given integrable model it provides a canonical way to exhibit the conservation laws (integrals of motion) which is usually the first step in constructing explicit solutions of the corresponding equations of motion. Indeed, the one-parameter family of flat connections allows one to define the monodromy matrix $T(z)$ which is the path-ordered exponential of the Lax component $L_\sigma(z)$

$$T(z) = \overleftarrow{\exp} \int_0^{2\pi} \mathrm{d}\sigma L_\sigma(z). \tag{1.91}$$

For definiteness, we assume that a model is defined on a circle $0 \leqslant \sigma < 2\pi$ and all dynamical variables are periodic functions of σ.

Let us derive the evolution equation for this matrix with respect to the parameter τ. We have

$$\begin{aligned}\partial_\tau T(z) &= \int_0^{2\pi} \mathrm{d}\sigma \left[\overleftarrow{\exp} \int_\sigma^{2\pi} L_\sigma\right] \partial_\tau L_\sigma \left[\overleftarrow{\exp} \int_0^\sigma L_\sigma\right] \\ &= \int_0^{2\pi} \mathrm{d}\sigma \left[\overleftarrow{\exp} \int_\sigma^{2\pi} L_\sigma\right] (\partial_\sigma L_\tau + [L_\tau, L_\sigma]) \left[\overleftarrow{\exp} \int_0^\sigma L_\sigma\right],\end{aligned}$$

where in the last formula we used the flatness of L_α. Now we note that the integrand of the expression above is the total derivative

$$\partial_\tau T(z) = \int_0^{2\pi} \mathrm{d}\sigma\, \partial_\sigma \left[\left(\overleftarrow{\exp} \int_\sigma^{2\pi} L_\sigma\right) L_\tau \left(\overleftarrow{\exp} \int_0^\sigma L_\sigma\right)\right]. \tag{1.92}$$

Given that the Lax connection is a periodic function of σ, for the monodromy we find the following evolution equation:

$$\partial_\tau T(z) = [L_\tau(0, \tau, z), T(z)]. \tag{1.93}$$

This important formula implies that the eigenvalues of $T(z)$ defined by the characteristic equation

$$\Gamma(z, \mu) \equiv \det(T(z) - \mu \mathbb{1}) = 0 \tag{1.94}$$

do not depend on τ, in other words they are integrals of motion. Thus, the spectral properties of the model are encoded into the monodromy matrix. Equation (1.94) defines an algebraic curve in $\mathbb{C}^2$ called the spectral curve.

An alternative way to obtain the evolution equation (1.93) is to notice that $T(\tau)$ introduced above represents the monodromy of a solution of the fundamental linear problem

$$\Psi(2\pi, \tau) = T(\tau)\Psi(0, \tau).$$

Indeed, if we differentiate this equation with respect to τ, we get

$$\partial_\tau \Psi(2\pi, \tau) = \partial_\tau T(\tau)\Psi(0, \tau) + T(\tau)\partial_\tau \Psi(0, \tau),$$

which, according to the fundamental linear system, gives

$$L_\tau(2\pi, \tau)T(\tau)\Psi(0, \tau) = \partial_\tau T(\tau)\Psi(0, \tau) + T(\tau)L_\tau(0, \tau)\Psi(0, \tau).$$

This relation implies the evolution equation (1.93).

An example: principal chiral model. To familiarize the reader with the concept of integrability, we consider, as an example, the so-called principal chiral model. This integrable system is rather similar but much simpler than the string sigma model introduced in the previous section and, therefore, our discussion here will provide a necessary warm-up before an actual handling of string integrability.

The principal chiral model is a nonlinear sigma model based on a field $\mathfrak{g} \equiv \mathfrak{g}(\sigma, \tau)$ with values in a Lie group. The action reads

$$S = -\frac{1}{2}\int \mathrm{d}\tau\, \mathrm{d}\sigma\, \gamma^{\alpha\beta}\, \mathrm{tr}(\partial_\alpha \mathfrak{g}\mathfrak{g}^{-1}\partial_\beta \mathfrak{g}\mathfrak{g}^{-1}),$$

where $\gamma^{\alpha\beta}$ is the Weyl-invariant metric introduced in section 1.2.1. Equations of motion are

$$\partial_\alpha(\gamma^{\alpha\beta}\partial_\beta \mathfrak{g}\mathfrak{g}^{-1}) = \partial_\alpha(\gamma^{\alpha\beta}\mathfrak{g}^{-1}\partial_\beta \mathfrak{g}) = 0, \tag{1.95}$$

and they can be conveniently written in terms of the left and right currents

$$A_l^\alpha = -\gamma^{\alpha\beta}\partial_\beta \mathfrak{g}\mathfrak{g}^{-1}, \qquad A_r^\alpha = -\gamma^{\alpha\beta}\mathfrak{g}^{-1}\partial_\beta \mathfrak{g} \tag{1.96}$$

as

$$\partial_\alpha A_l^\alpha = 0 = \partial_\alpha A_r^\alpha.$$

One can easily see that A_l and A_r are the Noether currents corresponding to multiplications of $\mathfrak{g}$ by a constant group element from the left $\mathfrak{g} \to \mathfrak{h}\mathfrak{g}$ and from the right $\mathfrak{g} \to \mathfrak{g}\mathfrak{h}$, respectively. These shifts from the left and from the right constitute the global symmetries of the model.

Introduce the following connection:

$$L_\alpha = \ell_1 A_\alpha + \ell_2 \gamma_{\alpha\beta}\epsilon^{\beta\rho}A_\rho, \tag{1.97}$$

where ℓ_1 and ℓ_2 are two undetermined parameters and A is either A^r or A^l. In two dimensions the zero-curvature condition (1.89) can be equivalently written as

$$2\epsilon^{\alpha\beta}\partial_\alpha L_\beta - \epsilon^{\alpha\beta}[L_\alpha, L_\beta] = 0. \tag{1.98}$$

Now we want to determine the coefficients ℓ_1 and ℓ_2 by requiring the fulfilment of equation (1.98) on-shell. Taking into account the following identity:

$$\epsilon^{\alpha\beta}\gamma_{\beta\rho}\epsilon^{\rho\delta} = \gamma^{\alpha\delta}, \tag{1.99}$$

a simple computation reveals that equation (1.98) for the connection (1.97) reduces to

$$2\ell_1\epsilon^{\alpha\beta}\partial_\alpha A_\beta - \left(\ell_1^2 - \ell_2^2\right)\epsilon^{\alpha\beta}[A_\alpha, A_\beta] + 2\ell_2\partial_\alpha A^\alpha = 0.$$

The last term vanishes due to the equations of motion. As to the first two terms, we recall that both A^r or A^l are flat, i.e.

$$\partial_\alpha A_\beta - \partial_\beta A_\alpha \pm [A_\alpha, A_\beta] = 0,$$

where the minus in front of the commutator term is for the right current and the plus for the left one, respectively. Thus, the first two terms will vanish due to the flatness of A provided the parameters ℓ are related as

$$\begin{aligned} &\ell_1^2 - \ell_2^2 - \ell_1 = 0 \qquad \text{for} \quad A = A^r, \\ &\ell_1^2 - \ell_2^2 + \ell_1 = 0 \qquad \text{for} \quad A = A^l. \end{aligned} \tag{1.100}$$

Both equations can be resolved in term of a single free parameter z so that we find two Lax formulations of the equations of motion of the principal chiral model

$$\begin{aligned} L_\alpha^r &= -\frac{z^2}{1-z^2}A_\alpha^r + \frac{z}{1-z^2}\gamma_{\alpha\beta}\epsilon^{\beta\rho}A_\rho^r, \\ L_\alpha^l &= \frac{z^2}{1-z^2}A_\alpha^l + \frac{z}{1-z^2}\gamma_{\alpha\beta}\epsilon^{\beta\rho}A_\rho^l. \end{aligned} \tag{1.101}$$

The parameter z plays now the role of the spectral parameter of the model. Both Lax connections exhibit first-order poles at $z = \pm 1$. These poles play a very special role. As we will see later, expanding the trace of the monodromy matrix around these poles leads to an explicit construction of *local* conserved charges. We also note that the leading term in the expansion of L's around $z = \infty$ ($z = 0$) coincides with the Noether current (the Hodge dual of the Noether current) corresponding to right or left global symmetries of the model. Finally, we remark that the connections L^r and L^l are related by the gauge transformation

$$L^l = hL^r h^{-1} + \mathrm{d}h h^{-1}$$

with $h = \mathfrak{g}$, which means that they are essentially equivalent.

Now we turn our attention to the construction of the Lax representation for our string sigma model.

1.3.2. Lax pair. To build up the zero-curvature representation of the string equations of motion, we start with the following ansatz for the Lax connection L_α:

$$L_\alpha = \ell_0 A_\alpha^{(0)} + \ell_1 A_\alpha^{(2)} + \ell_2 \gamma_{\alpha\beta}\epsilon^{\beta\rho} A_\rho^{(2)} + \ell_3 A_\alpha^{(1)} + \ell_4 A_\alpha^{(3)}, \tag{1.102}$$

where ℓ_i are undetermined constants and $A^{(k)}$ are the $\mathbb{Z}_4$-components of the flat connection (1.33). The connection L_α is required to have zero curvature (1.89) as a consequence of the dynamical equations (1.48) and the flatness of A_α. This requirement will impose certain constraints on ℓ_i, much similar to the case of the principal chiral model discussed in the previous section.

Computing the curvature of L_α, we expand the resulting expression into the sum of the homogeneous components $\mathscr{G}^{(k)}$ under the $\mathbb{Z}_4$-grading. First, the projection on $\mathscr{G}^{(0)}$ reads

$$2\ell_0\epsilon^{\alpha\beta}\partial_\alpha A_\beta^{(0)} - \epsilon^{\alpha\beta}\left(\ell_0^2\big[A_\alpha^{(0)}, A_\beta^{(0)}\big] + \left(\ell_1^2 - \ell_2^2\right)\big[A_\alpha^{(2)}, A_\beta^{(2)}\big] + 2\ell_3\ell_4\big[A_\alpha^{(1)}, A_\beta^{(3)}\big]\right) = 0.$$

The flatness of $A^{(0)}$ then implies

$$\ell_0 = 1, \qquad \ell_1^2 - \ell_2^2 = 1, \qquad \ell_3\ell_4 = 1. \tag{1.103}$$

Second, for the projection on $\mathscr{G}^{(2)}$ we find

$$\begin{aligned}&\ell_1\epsilon^{\alpha\beta}\partial_\alpha A_\beta^{(2)} + \ell_2\partial_\alpha\left(\gamma^{\alpha\beta}A_\beta^{(2)}\right) - (\epsilon^{\alpha\beta}\ell_0\ell_1 + \gamma^{\alpha\beta}\ell_0\ell_2)\big[A_\alpha^{(0)}, A_\beta^{(2)}\big]\\ &\qquad - \tfrac{1}{2}\epsilon^{\alpha\beta}\ell_3^2\big[A_\alpha^{(1)}, A_\beta^{(1)}\big] - \tfrac{1}{2}\epsilon^{\alpha\beta}\ell_4^2\big[A_\alpha^{(3)}, A_\beta^{(3)}\big] = 0.\end{aligned}$$

Using the flatness condition for $A^{(2)}$, we can bring this equation to the form

$$\partial_\alpha\left(\gamma^{\alpha\beta}A_\beta^{(2)}\right) - \gamma^{\alpha\beta}\big[A_\alpha^{(0)}, A_\beta^{(2)}\big] - \epsilon^{\alpha\beta}\frac{\left(\ell_3^2 - \ell_1\right)}{2\ell_2}\big[A_\alpha^{(1)}, A_\beta^{(1)}\big] - \epsilon^{\alpha\beta}\frac{\left(\ell_4^2 - \ell_1\right)}{2\ell_2}\big[A_\alpha^{(3)}, A_\beta^{(3)}\big] = 0.$$

The last expression coincides with the string equations of motion (1.49) provided

$$\frac{\ell_3^2 - \ell_1}{\ell_2} = -\kappa, \qquad \frac{\ell_4^2 - \ell_1}{\ell_2} = \kappa. \tag{1.104}$$

Third, projections on $\mathscr{G}^{(1)}$ and $\mathscr{G}^{(3)}$ are

$$\ell_3\epsilon^{\alpha\beta}\partial_\alpha A_\beta^{(1)} - \epsilon^{\alpha\beta}\ell_0\ell_3\big[A_\alpha^{(0)}, A_\beta^{(1)}\big] - \epsilon^{\alpha\beta}\ell_1\ell_4\big[A_\alpha^{(2)}, A_\beta^{(3)}\big] + \gamma^{\alpha\beta}\ell_2\ell_4\big[A_\alpha^{(2)}, A_\beta^{(3)}\big] = 0,$$

$$\ell_4\epsilon^{\alpha\beta}\partial_\alpha A_\beta^{(3)} - \epsilon^{\alpha\beta}\ell_0\ell_4\big[A_\alpha^{(0)}, A_\beta^{(3)}\big] - \epsilon^{\alpha\beta}\ell_1\ell_3\big[A_\alpha^{(2)}, A_\beta^{(1)}\big] + \gamma^{\alpha\beta}\ell_2\ell_3\big[A_\alpha^{(2)}, A_\beta^{(1)}\big] = 0.$$

Once again, by invoking the flatness of $A^{(1,3)}$, we can rewrite these equations as follows:

$$\left(\gamma^{\alpha\beta} - \frac{\ell_1\ell_4 - \ell_3}{\ell_2\ell_4}\epsilon^{\alpha\beta}\right)\big[A_\alpha^{(2)}, A_\beta^{(3)}\big] = 0,$$

$$\left(\gamma^{\alpha\beta} + \frac{\ell_4 - \ell_1\ell_3}{\ell_2\ell_3}\epsilon^{\alpha\beta}\right)\big[A_\alpha^{(2)}, A_\beta^{(1)}\big] = 0.$$

These will coincide with the string equations (1.52) provided

$$\frac{\ell_1\ell_4-\ell_3}{\ell_2\ell_4}=\kappa, \qquad \frac{\ell_4-\ell_1\ell_3}{\ell_2\ell_3}=\kappa. \tag{1.105}$$

Summing up equations (1.104), we find

$$2\ell_1=\ell_3^2+\ell_4^2. \tag{1.106}$$

The same condition follows from equations (1.105) upon taking into account that $\ell_3\ell_4=1$. Furthermore, it appears that the parameter κ gets fixed up to the sign. Indeed, multiplying equations (1.105) and using equations (1.103), (1.106), we get

$$\kappa^2=1, \tag{1.107}$$

which is precisely the condition for having κ-symmetry! Thus, we have obtained a striking result: integrability of the equations of motion for the Lagrangian (1.35), i.e. existence of the corresponding Lax connection, implies that the model possesses κ-symmetry.

Proceeding, we uniformize the parameters ℓ_i in terms of a single variable z taking values on the Riemann sphere

$$\begin{aligned} &\ell_0=1, \qquad \ell_1=\frac{1}{2}\left(z^2+\frac{1}{z^2}\right),\\ &\ell_2=-\frac{1}{2\kappa}\left(z^2-\frac{1}{z^2}\right), \qquad \ell_3=z, \qquad \ell_4=\frac{1}{z}. \end{aligned} \tag{1.108}$$

The reader can easily verify that these ℓ_i solve all the constraints imposed by the zero curvature for L_α. For a given $\kappa=\pm1$, there is a unique Lax connection which is a meromorphic matrix-valued function on the Riemann sphere.

Finally, we point out how the grading map Ω acts on the Lax connection L_α. Since Ω is an automorphism of $\mathfrak{sl}(4|4)$, the curvature of $\Omega(L_\alpha)$ also vanishes. It can be easily checked that $\Omega(L_\alpha)$ is related to L_α by a certain diffeomorphism of the spectral parameter, namely,

$$\Omega(L_\alpha(z))=L_\alpha(iz),$$

i.e. z undergoes a rotation by the angle $\pi/2$. Using the explicit form of Ω, we can write the last relation as

$$\mathcal{K}L_\alpha^{\rm st}(z)\mathcal{K}^{-1}=-L_\alpha(iz). \tag{1.109}$$

Since z is complex, the Lax connection takes values in $\mathfrak{sl}(4|4)$ rather than in $\mathfrak{su}(2,2|4)$. Obviously, the action of Ω on L_α is compatible with the fact that Ω is the fourth-order automorphism of $\mathfrak{sl}(4|4)$.

Finally, we mention the action of the parity transformation (1.67) on the Lax connection. Under $\sigma\to-\sigma$ we have $A_\tau\to A_\tau$ and $A_\sigma\to-A_\sigma$. Thus,

$$\mathscr{P}L_\tau(z)|_{\sigma\to-\sigma}\mathscr{P}^{-1}=L_\tau^{\mathscr{P}}(1/z), \qquad \mathscr{P}L_\sigma(z)|_{\sigma\to-\sigma}\mathscr{P}^{-1}=-L_\sigma^{\mathscr{P}}(1/z),$$

where we have taken into account that the parity transformation exchanges[14] the projections $A^{(1)}$ and $A^{(3)}$. Here $L_\alpha^{\mathscr{P}}$ is the same connection (1.102) where $\mathfrak{g}(\sigma)$ in $A_\alpha=-\mathfrak{g}^{-1}\partial_\alpha\mathfrak{g}$ is replaced by $\mathfrak{g}^{\mathscr{P}}(\sigma)=\mathfrak{g}(-\sigma)$. Obviously, $L_\alpha^{\mathscr{P}}$ retains vanishing curvature.

In summary, we have shown that the string equations of motion admit zero-curvature representation which ensures their kinematical integrability. We have also seen that inclusion of the Wess–Zumino term into the string Lagrangian is allowed by integrability only for $\kappa=\pm1$, i.e. only for those values of κ for which the model has κ-symmetry.

[14] Specifying an explicit dependence of the Lax connection on κ as $L_\alpha(z,\kappa)$, we see that without changing $\sigma\to-\sigma$ the connection enjoys the following property $\mathscr{P}L_\alpha(z,\kappa)\mathscr{P}^{-1}=L_\alpha(1/z,-\kappa)$.

1.3.3. Integrability and symmetries. In the previous section we have shown that string equations of motion admit the Lax representation provided the parameter κ in the Lagrangian takes values ± 1. It is for these values of κ that the model exhibits the local fermionic symmetry. In addition to the κ-symmetry, the string sigma model has the usual reparametrization invariance. Due to these local symmetries not all degrees of freedom appearing in the Lagrangian (1.35) are physical. Thus, ultimately we would like to understand if and how integrability is inherited by the physical subspace which is obtained by making a gauge choice and imposing the Virasoro constraints. In this section we will make a first step in this direction by analyzing in detail the transformation properties of the Lax connection under the κ-symmetry and diffeomorphism transformations. We also indicate a relation between the Lax connection and the global $\mathfrak{psu}(2,2|4)$ symmetry of the model.

We start with the analysis of the relationship between the Lax connection and κ-symmetry. Recalling equations (1.72) which describe how the $\mathbb{Z}_4$-components of A transform under κ-symmetry, it is straightforward to find[15]

$$\delta L_\alpha = [L_\alpha, \Lambda] - \partial_\alpha\Lambda + (\ell_4 - \ell_1\ell_3)\big[A^{(2)}_\alpha, \epsilon^{(1)}\big] - \ell_2\ell_3\big[\epsilon_{\alpha\beta}\gamma^{\beta\delta}A^{(2)}_\delta, \epsilon^{(1)}\big] \\ + \big[\big(\ell_1 - \ell_3^2\big)A^{(1)}_\alpha + \ell_2\epsilon_{\alpha\beta}\gamma^{\beta\delta}A^{(1)}_\delta, \epsilon^{(1)}\big] + \ell_2\epsilon_{\alpha\beta}\delta\gamma^{\beta\delta}A^{(2)}_\delta.$$

Here $\Lambda = \ell_3\epsilon^{(1)}$. Taking into account the relations between the coefficients ℓ_i found in the previous section from the requirement of integrability, the last formula can be cast into the form

$$\delta L_\alpha = [L_\alpha, \Lambda] - \partial_\alpha\Lambda + \ell_2\ell_3\epsilon_{\alpha\beta}[A^{(2),\beta}_-, \epsilon^{(1)}] + \ell_2\epsilon_{\alpha\beta}\big(2\big[A^{(1),\beta}_+, \epsilon^{(1)}\big] + \delta\gamma^{\beta\delta}A^{(2)}_\delta\big).$$

The Λ-dependent term here is nothing else but an infinitesimal gauge transformation of the Lax connection. Under this transformation, the curvature of the transformed connection retains its zero value. On the other hand, the last two terms proportional to $\ell_2\ell_3$ and ℓ_2 would destroy the zero-curvature condition unless they (separately) vanish. It turns out that vanishing of these terms is equivalent to the requirement of the Virasoro constraints as well as equations of motion for the fermions! Consider the first term

$$I_1 \equiv \big[A^{(2)}_{\alpha,-}, \epsilon^{(1)}\big] = \big[A^{(2)}_{\alpha,-}, A^{(2)}_{\beta,-}\kappa^{(1),\beta}_+ + \kappa^{(1),\beta}_+ A^{(2)}_{\beta,-}\big] = \tfrac{1}{8}\,\mathrm{str}\big(A^{(2)}_{\alpha,-}A^{(2)}_{\beta,-}\big)\big[\Upsilon, \kappa^{(1),\beta}_+\big].$$

Here we used equation (1.80) and also equation (1.76) stating that $A^{(2)}_{\alpha,-}$ and $A^{(2)}_{\beta,-}$ with different α and β are proportional to each other. It is not hard to prove that

$$\mathrm{str}\big(A^{(2)}_{\alpha,-}A^{(2)}_{\beta,-}\big) = 0$$

implies fulfilment of the Virasoro constraint (1.57) and vice versa.

The second unwanted term involves an expression

$$I_2 \equiv \big[A^{(1),\alpha}_+, \epsilon^{(1)}\big] = \big[A^{(1),\alpha}_+, A^{(2)}_{\beta,-}\kappa^{(1),\beta}_+ + \kappa^{(1),\beta}_+ A^{(2)}_{\beta,-}\big] = \big[A^{(1),\beta}_+, A^{(2)}_{\beta,-}\kappa^{(1),\alpha}_+ + \kappa^{(1),\alpha}_+ A^{(2)}_{\beta,-}\big],$$

where we made use of the identity (1.81). Taking into account the equation of motion for fermions, $\big[A^{(1),\beta}_+, A^{(2)}_{\beta,-}\big] = 0$, the last formula boils down to

$$I_2 = A^{(2)}_{\beta,-}\big[A^{(1),\beta}_+, \kappa^{(1),\alpha}_+\big] + \big[A^{(1),\beta}_+, \kappa^{(1),\alpha}_+\big]A^{(2)}_{\beta,-}.$$

Since the commutator $\big[A^{(1),\beta}_+, \kappa^{(1),\alpha}_+\big]$ belongs to the space $\mathscr{G}^{(2)}$, we can write

$$\big[A^{(1),\beta}_+, \kappa^{(1),\alpha}_+\big] = \begin{pmatrix} m^a\gamma^a & 0 \\ 0 & n^a\gamma^a \end{pmatrix} + \frac{1}{8}\,\mathrm{str}\big(\Upsilon\big[A^{(1),\beta}_+, \kappa^{(1),\alpha}_+\big]\big)\mathbb{1},$$

for some coefficients m^a and n^a. Therefore,

$$I_2 = \left\{A^{(2)}_{\beta,-}, \begin{pmatrix} m^a\gamma^a & 0 \\ 0 & n^a\gamma^a \end{pmatrix}\right\} + \frac{1}{4}\,\mathrm{str}\big(\Upsilon\big[A^{(1),\beta}_+, \kappa^{(1),\alpha}_+\big]\big)A^{(2)}_{\beta,-}$$

[15] For our present purposes it is enough to consider a variation with non-trivial $\epsilon^{(1)}$ only.

Expanding the element $A^{(2)}_{\beta,-}$ over the Dirac matrices and substituting it in the anti-commutator above, we see that, due to the Clifford algebra, $2I_2$ must have the following structure

$$2I_2 = \rho_1 \mathbb{1} + \rho_2 \Upsilon - \tfrac{1}{2}\,\mathrm{str}\big(\Upsilon\big[\kappa_+^{(1),\alpha}, A_+^{(1),\beta}\big]\big)A^{(2)}_{\beta,-}. \tag{1.110}$$

Actually, the coefficient ρ_2 must vanish as the supertrace of I_2 equals zero (this follows from the original definition of I_2 as a commutator of two terms). The last term proportional to $A^{(2)}_{\beta,-}$ will then cancel in equation (1.110) the term containing the variation of the metric $\delta\gamma^{\alpha\beta}A^{(2)}_{\beta} = \delta\gamma^{\alpha\beta}A^{(2)}_{\beta,-}$. Finally, the term in equation (1.110) proportional to the identity matrix is unessential because the Lax representation is understood modulo an element $\mathrm{i}\mathbb{1}$.

Thus, we have obtained the following important result. Although the Virasoro constraints (1.57) do not apparently follow from the zero-curvature condition, we see that upon κ-symmetry transformations the Lax connection retains its zero curvature if and only if the Virasoro constraints (and equations of motion for fermions) are satisfied. This shows that the local symmetries of the model and the existence of the Lax connection are tightly related to each other.

Let us now show that diffeomorphisms also induce gauge transformations of the Lax connection. Indeed, under a diffeomorphism $\sigma^\alpha \to \sigma^\alpha + f^\alpha(\sigma)$ any 1-form and, in particular, the Lax connection transforms as follows:

$$\delta L_\alpha = f^\beta\partial_\beta L_\alpha + L_\beta\partial_\alpha f^\beta. \tag{1.111}$$

Using the zero-curvature condition for L_α, we can rewrite the last formula as

$$\delta L_\alpha = f^\beta(\partial_\alpha L_\beta + [L_\beta, L_\alpha]) + L_\beta\partial_\alpha f^\beta = \partial_\alpha(f^\beta L_\beta) + [f^\beta L_\beta, L_\alpha], \tag{1.112}$$

which is a gauge transformation with aparameter $f^\beta L_\beta$.

Now we explain the interrelation between the Lax connection and the generators of the global $\mathfrak{psu}(2,2|4)$ symmetry. So far our discussion of the Lax connection was based on the 1-form $A = -\mathrm{d}\mathfrak{g}\mathfrak{g}^{-1}$ which, as the reader undoubtedly noted, is analogous to the right connection of the principal chiral model. At $z = 1$ the Lax connection (1.108) turns into A_α. As we have already mentioned, the condition of zero curvature (1.89) is invariant with respect to the gauge transformations

$$L \to L' = hLh^{-1} + \mathrm{d}hh^{-1}.$$

The inhomogeneous term on the right-hand side does not depend on z and, therefore, this gauge freedom can be used to gauge away the constant piece A arising at $z = 1$. For this one has to take $h = \mathfrak{g}$. Indeed, define $a^{(i)} = \mathfrak{g}A^{(i)}\mathfrak{g}^{-1}$ and represent the 'dual' 1-form $\tilde{A} = -\mathrm{d}\mathfrak{g}\mathfrak{g}^{-1}$ in the following way:

$$\tilde{A} = \mathfrak{g}A\mathfrak{g}^{-1} = \mathfrak{g}(A^{(0)} + A^{(1)} + A^{(2)} + A^{(3)})\mathfrak{g}^{-1} = a^{(0)} + a^{(1)} + a^{(2)} + a^{(3)}.$$

Then the result of the gauge transformation of L takes the form

$$L_\alpha = \ell_0 a^{(0)}_\alpha + \ell_1 a^{(2)}_\alpha + \ell_2\gamma_{\alpha\beta}\epsilon^{\beta\rho}a^{(2)}_\rho + \ell_3 a^{(1)}_\alpha + \ell_4 a^{(3)}_\alpha, \tag{1.113}$$

where $\ell_0 = 0$ and the other coefficients ℓ_i are given by

$$\ell_1 = \frac{(1-z^2)^2}{2z^2}, \qquad \ell_2 = -\frac{1}{2\kappa}\left(z^2 - \frac{1}{z^2}\right), \qquad \ell_3 = z - 1, \qquad \ell_4 = \frac{1}{z} - 1.$$

Expanding this connection around $w = 1 - z$

$$L_\alpha = \frac{2w}{\kappa}\mathcal{L}_\alpha + \cdots, \tag{1.114}$$

we discover that the leading term $\mathcal{L}_\alpha$ is

$$\mathcal{L}_\alpha = \gamma_{\alpha\beta}\epsilon^{\beta\rho}a^{(2)}_\rho - \frac{\kappa}{2}\big(a^{(1)}_\alpha - a^{(3)}_\alpha\big).$$

The zero-curvature condition is satisfied at each order in w; at first order in w it gives

$$\partial_\alpha \mathcal{L}_\beta - \partial_\beta \mathcal{L}_\alpha = 0 \quad \Longrightarrow \quad \partial_\alpha(\epsilon^{\alpha\beta}\mathcal{L}_\beta) = 0,$$

which is obviously the conservation law for a non-Abelian current

$$J^\alpha = g\epsilon^{\alpha\beta}\mathcal{L}_\beta = g\left[\gamma^{\alpha\beta}a_\beta^{(2)} - \frac{\kappa}{2}\epsilon^{\alpha\beta}\left(a_\beta^{(1)} - a_\beta^{(3)}\right)\right]. \tag{1.115}$$

Comparing the last expression with equations (1.45), (1.54), we conclude that J^α is just the Noether current corresponding to the global $\mathfrak{psu}(2,2|4)$ symmetry of the model. The dual 1-form $\tilde{A}$ is an analog of the left connection of the principal chiral model.

One can analyze the expansion of the Lax connection around $z=-1$ in a similar fashion. Expanded around $z=-1$, the connection exhibits a constant piece which can be gauged away by a proper gauge transformation. After this is done, at order $(z+1)$ one finds a non-Abelian conserved current, which is again the Noether current generating the global $\mathfrak{psu}(2,2|4)$-symmetry.

1.4. Coset parametrizations

This section is devoted to the discussion of various embeddings of the coset space (1.1) into the supergroup SU(2, 2|4). We put an emphasis on a particular embedding which is most suitable for the light-cone gauge fixing. We also identify a bosonic subalgebra of the full symmetry algebra which acts linearly on the coordinates of the coset space.

1.4.1. Coset parametrization. To give an explicit expression for the Lagrangian density (1.35) in terms of the coset degrees of freedom varying over the two-dimensional world-sheet, it is necessary to choose an embedding of the coset element (1.1) into the supergroup SU(2, 2|4). This can be done in many different ways, all of them are related by nonlinear field redefinitions. Before we motivate our preferred coset parametrization, we need to describe the space $\mathscr{G}^{(2)}$ constituting the orthogonal complement of $\mathfrak{so}(4,1)\oplus\mathfrak{so}(5)$ in the bosonic subalgebra of $\mathfrak{psu}(2,2|4)$.

The space $\mathscr{G}^{(2)} \subset \mathscr{G} = \mathfrak{psu}(2,2|4)$ is spanned by solutions to the following equation:

$$\mathcal{K}M^{\rm st}\mathcal{K}^{-1} = M,$$

which for M even is equivalent to

$$M^t = \mathcal{K}M\mathcal{K}^{-1}.$$

According to the discussion of section 1.1.2, the matrices $M = M^{(2)}$ solving the equation above can be parametrized as

$$M = \frac{1}{2}\begin{pmatrix} {\rm i}t\gamma^5 + z^i\gamma^i & 0 \\ 0 & {\rm i}\phi\gamma^5 + {\rm i}y^i\gamma^i \end{pmatrix}, \tag{1.116}$$

where the summation index i runs from 1 to 4. As will be explained in appendix 1.5.1, the coordinates t, z^i cover the AdS_5 space, while ϕ, y^i cover the 5-sphere. In particular, ϕ parametrizes a big circle in S^5 and it has range $0 \leqslant \phi < 2\pi$. More generally, since we deal with closed strings, the global coordinates on the sphere must be periodic functions of σ. On the other hand, ϕ is an angle and, therefore, one can have configurations with a non-trivial winding

$$\phi(2\pi) - \phi(0) = 2\pi m, \tag{1.117}$$

where m is an integer. All the coordinates are assumed to be periodic functions of σ (we do not allow winding in the time direction).

One obvious way to define an embedding of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ space into the bosonic subgroup of SU(2, 2|4) is just to exponentiate an element (1.116)

$$\mathfrak{g}_{\mathfrak{b}} = \exp \frac{1}{2} \begin{pmatrix} \mathrm{i}t\gamma^5 + z^i\gamma^i & 0 \\ 0 & \mathrm{i}\phi\gamma^5 + \mathrm{i}y^i\gamma^i \end{pmatrix}.$$

The fermionic degrees of freedom can be incorporated in the following group element:

$$\mathfrak{g}_{\mathfrak{f}} = \exp \chi, \ \chi = \begin{pmatrix} 0 & \Theta \\ -\Theta^\dagger\Sigma & 0 \end{pmatrix}. \tag{1.118}$$

A group element describing an embedding of the coset space (1.1) into SU(2, 2|4) can be then constructed as

$$\mathfrak{g} = \mathfrak{g}_{\mathfrak{f}}\mathfrak{g}_{\mathfrak{b}}. \tag{1.119}$$

Clearly, this is just one of infinitely many ways to choose a coset representative; for instance, one could also define $\mathfrak{g} = \mathfrak{g}_{\mathfrak{b}}\mathfrak{g}_{\mathfrak{f}}$.

It appears, however, that the choice (1.119) is particularly convenient to manifest the global bosonic symmetries of the model, because the latter act linearly on fermionic variables. Indeed, the symmetry group acts on a coset element by multiplication from the left, see equations (1.41) and (1.42). If a coset element is realized as in equation (1.119), then the action of $G \in \mathrm{SU}(2,2) \times \mathrm{SU}(4)$ preserves the structure of the fermionic coset representative

$$G \cdot \mathfrak{g} = G\mathfrak{g}_{\mathfrak{f}}G^{-1} \cdot G\mathfrak{g}_{\mathfrak{b}} = G\mathfrak{g}_{\mathfrak{f}}G^{-1} \cdot \mathfrak{g}'_{\mathfrak{b}}\mathfrak{h},$$

where $\mathfrak{h}$ is a compensating element from SO(4, 1) × SO(5). From here we deduce that the matrix $\mathfrak{g}_{\mathfrak{f}}$ transforms as

$$\mathfrak{g}_{\mathfrak{f}} \to G\mathfrak{g}_{\mathfrak{f}}G^{-1} = \exp G\chi G^{-1}.$$

Thus, fermions undergo the adjoint (linear) action of G, while bosons generically transform in a nonlinear fashion: $\mathfrak{g}_{\mathfrak{b}} \to \mathfrak{g}'_{\mathfrak{b}}$. In particular, fermions are charged under *all* Cartan generators of $\mathfrak{psu}(2,2|4)$, the latter represents a set of commuting $\mathfrak{u}(1)$-isometries of the coset space (1.1).

Another reason to choose a coset representative (1.119) is that in this case supersymmetry transformations act on the fermionic and bosonic variables in a simple way. Indeed, under an infinitesimal supersymmetry transformation with a fermionic parameter ϵ the coset variables undergo the following transformation:

$$\delta_\epsilon\chi = \epsilon, \qquad \delta_\epsilon\mathfrak{g}_{\mathfrak{b}} = \tfrac{1}{2}[\epsilon, \chi]\mathfrak{g}_{\mathfrak{b}} - \mathfrak{g}_{\mathfrak{b}}\mathfrak{h}, \tag{1.120}$$

where $\mathfrak{h}$ is a compensating element from SO(4, 1) × SO(5). The last formula makes it obvious that invariance of the model under supersymmetry transformations requires fermionic variables in the representation (1.119) to be periodic functions of σ.

As will be discussed in the following section, fixing the light-cone gauge is greatly facilitated by working with fermions which are *neutral* under the isometries corresponding to shifts of the AdS time t and the sphere angle ϕ. By the above, fermions of the coset element (1.119) do not meet this requirement. The idea is, therefore, to redefine the original fermionic variables in such a fashion that they become neutral under the isometries related to t and ϕ. This can be understood in the following way. Introduce a diagonal matrix

$$\Lambda(t,\phi) = \exp \begin{pmatrix} \frac{\mathrm{i}}{2}t\gamma^5 & 0 \\ 0 & \frac{\mathrm{i}}{2}\phi\gamma^5 \end{pmatrix} \tag{1.121}$$

with the property $\Lambda(t_1 + t_2, \phi_1 + \phi_2) = \Lambda(t_1, \phi_1)\Lambda(t_2, \phi_2)$, and the following exponential:

$$\mathfrak{g}(\mathbb{X}) = \exp \mathbb{X}, \qquad \mathbb{X} = \begin{pmatrix} \frac{1}{2}z^i\gamma^i & 0 \\ 0 & \frac{\mathrm{i}}{2}y^i\gamma^i \end{pmatrix}. \tag{1.122}$$

Now, instead of (1.119), consider a new parametrization of the coset representative

$$\mathfrak{g} = \Lambda(t,\phi)\mathfrak{g}(\chi)\mathfrak{g}(\mathbb{X}), \tag{1.123}$$

where $\mathfrak{g}(\chi) \equiv \mathfrak{g}_{\mathfrak{f}}$. Obviously, an element G corresponding to global shifts $t \to t+a$, $\phi \to \phi+b$ can be identified with $\Lambda(a,b)$. Thus, under the left multiplication

$$G \cdot \mathfrak{g} = \Lambda(a,b)\Lambda(t,\phi)\mathfrak{g}(\chi)\mathfrak{g}(\mathbb{X}) = \Lambda(t+a,\phi+b)\mathfrak{g}(\chi)\mathfrak{g}(\mathbb{X}), \tag{1.124}$$

i.e. both χ and $\mathbb{X}$ remain untouched by this transformation. In other words, with our new choice (1.123), not only the fermions χ but also all the remaining eight bosons z^i and y^i, appear to be neutral under the isometries related to t and ϕ ! This property motivates our choice (1.123). In fact, coset representatives (1.119) and (1.123) are related to each other by a nonlinear field redefinition, which for fermionic variables is of the form $\chi \to \Lambda(t,\phi)\chi\Lambda(t,\phi)^{-1}$.

It should be noted, however, that nonlinear field redefinitions can change the boundary conditions for the world-sheet fields. In parametrization (1.119) fermions χ transform linearly under all bosonic symmetries and they are periodic functions of σ. To pass to parametrization (1.123), we redefine

$$\chi \to \chi' = \Lambda^{-1}\chi\Lambda \quad\Longrightarrow\quad \Theta \to \Theta' = \mathrm{e}^{\frac{\mathrm{i}}{2}(\phi-t)\gamma_5}\Theta, \tag{1.125}$$

where we have invoked the parametrization (1.118). As a result, the new fermions satisfy the following boundary conditions:

$$\Theta'(\sigma+2\pi) = \mathrm{e}^{\mathrm{i}\pi m\gamma_5}\Theta'(\sigma), \tag{1.126}$$

i.e. they remain periodic for m even (the even winding number sector) and they become anti-periodic for m odd (the odd winding number sector).

We conclude our discussion of coset representatives by emphasizing that given the structure (1.123), one is entirely free to choose parametrizations for $\mathfrak{g}(\chi)$ and $\mathfrak{g}(\mathbb{X})$ different from those in equations (1.118) and (1.122). In particular, in appendix 1.5.2 we describe another useful choice for the element $\mathfrak{g}(\mathbb{X})$.

1.4.2. Linearly realized bosonic symmetries. Well adjusted to the light-cone gauge, parametrization (1.123) does not allow for a linear realization of all the bosonic symmetries. Our next task is, therefore, to determine a maximal subgroup of the bosonic symmetry group which acts linearly on the dynamical fields $\mathbb{X}$ and χ. This subgroup will then coincide with the manifest bosonic symmetry of the light-cone gauge-fixed string Lagrangian.

It is easy to see that the centralizer of the $\mathfrak{u}(1)$-isometries corresponding to shifts of t and ϕ in the algebra $\mathfrak{su}(2,2) \oplus \mathfrak{su}(4)$ coincides with

$$\mathfrak{C} = \mathfrak{so}(4) \oplus \mathfrak{so}(4) = \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2), \tag{1.127}$$

where the first factor is $\mathfrak{so}(4) \subset \mathfrak{so}(4,1) \subset \mathfrak{so}(4,2)$ and the second one $\mathfrak{so}(4) \subset \mathfrak{so}(5) \subset \mathfrak{so}(6)$. Indeed, both copies of $\mathfrak{so}(4)$ are generated by $\frac{1}{4}[\gamma^i,\gamma^j], i,j = 1,\ldots,4$ because the latter matrices commute with $\mathrm{i}\gamma^5$ generating shifts in the t- or ϕ-directions. Let now G be a group element corresponding to a Lie algebra element from (1.127). Then $G\Lambda(t,\phi)G^{-1} = \Lambda(t,\phi)$. Due this condition, one gets

$$G \cdot \mathfrak{g} = \Lambda(t,\phi) \cdot G\mathfrak{g}(\chi)G^{-1} \cdot G\mathfrak{g}(\mathbb{X})G^{-1} \cdot G.$$

Now one can recognize that the last G in the right-hand side of this formula is nothing else but the compensating element $\mathfrak{h}$ from $\mathrm{SO}(4,1) \times \mathrm{SO}(5)$: $\mathfrak{h} = G$. Indeed, the adjoint transformation with G preserves the structure of the coset element $\mathfrak{g}(\mathbb{X})$, because the generator $\frac{1}{4}[\gamma^i,\gamma^j]$ commutes with γ^k for $j \neq k \neq i$ and is equal to $2\gamma^j$ for $k = j$. Thus, under the action of G both bosons and fermions undergo a linear transformation

$$\chi \to \chi' = G\chi G^{-1}, \qquad \mathbb{X} \to \mathbb{X}' = G\mathbb{X}G^{-1}.$$

To conclude, the centralizer $\mathfrak{C}$ of the isometries related to t and ϕ induces linear transformations of the dynamical variables.

In terms of 2 by 2 blocks a matrix G from the centralizer can be represented as follows:

$$G = \begin{pmatrix} \mathfrak{g}_1 & 0 & 0 & 0 \\ 0 & \mathfrak{g}_2 & 0 & 0 \\ 0 & 0 & \mathfrak{g}_3 & 0 \\ 0 & 0 & 0 & \mathfrak{g}_4 \end{pmatrix}. \tag{1.128}$$

Here $\mathfrak{g}_1, \ldots, \mathfrak{g}_4$ denote four independent copies of SU(2). Analogously, the elements $\mathbb{X}$ and χ can be represented as

$$\mathbb{X} = \begin{pmatrix} 0 & Z & 0 & 0 \\ Z^\dagger & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{i}Y \\ 0 & 0 & \mathrm{i}Y^\dagger & 0 \end{pmatrix}, \qquad \chi = \begin{pmatrix} 0 & 0 & \Theta_1 & \Theta_2 \\ 0 & 0 & \Theta_3^\dagger & \Theta_4 \\ -\Theta_1^\dagger & \Theta_3 & 0 & 0 \\ -\Theta_2^\dagger & \Theta_4^\dagger & 0 & 0 \end{pmatrix}. \tag{1.129}$$

Here Z and Y are two 2×2 matrices which incorporate eight bosonic degrees of freedom

$$Z = \frac{1}{2}\begin{pmatrix} z_3 - \mathrm{i}z_4 & -z_1 + \mathrm{i}z_2 \\ z_1 + \mathrm{i}z_2 & z_3 + \mathrm{i}z_4 \end{pmatrix}, \qquad Y = \frac{1}{2}\begin{pmatrix} y_3 - \mathrm{i}y_4 & -y_1 + \mathrm{i}y_2 \\ y_1 + \mathrm{i}y_2 & y_3 + \mathrm{i}y_4 \end{pmatrix}, \tag{1.130}$$

while four 2×2 blocks $\Theta_1, \ldots, \Theta_4$ comprise 16 complex fermions. Matrices Z and Y satisfy the following reality condition:

$$Z^\dagger = \epsilon Z^t \epsilon^{-1}, \qquad Y^\dagger = \epsilon Y^t \epsilon^{-1}, \qquad \epsilon \equiv \mathrm{i}\sigma_2, \tag{1.131}$$

where σ_2 is the Pauli matrix.

Thus, we deduce that under the action of G

$$\mathbb{X} \to G\mathbb{X}G^{-1} = \begin{pmatrix} 0 & \mathfrak{g}_1 Z \mathfrak{g}_2^{-1} & 0 & 0 \\ \mathfrak{g}_2 Z^\dagger \mathfrak{g}_1^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{i}\mathfrak{g}_3 Y \mathfrak{g}_4^{-1} \\ 0 & 0 & \mathrm{i}\mathfrak{g}_4 Y^\dagger \mathfrak{g}_3^{-1} & 0 \end{pmatrix} \tag{1.132}$$

and

$$\chi \to G\chi G^{-1} = \begin{pmatrix} 0 & 0 & \mathfrak{g}_1\Theta_1\mathfrak{g}_3^{-1} & \mathfrak{g}_1\Theta_2\mathfrak{g}_4^{-1} \\ 0 & 0 & \mathfrak{g}_2\Theta_3^\dagger\mathfrak{g}_3^{-1} & \mathfrak{g}_2\Theta_4\mathfrak{g}_4^{-1} \\ -\mathfrak{g}_3\Theta_1^\dagger\mathfrak{g}_1^{-1} & \mathfrak{g}_3\Theta_3\mathfrak{g}_2^{-1} & 0 & 0 \\ -\mathfrak{g}_4\Theta_2^\dagger\mathfrak{g}_1^{-1} & \mathfrak{g}_4\Theta_4^\dagger\mathfrak{g}_2^{-1} & 0 & 0 \end{pmatrix}. \tag{1.133}$$

Before we proceed with discussing these symmetry transformations, let us note that, according to equation (1.87), one can implement the κ-symmetry gauge by requiring the absence of fermionic blocks $\Theta_1, \Theta_1^\dagger$ and $\Theta_4, \Theta_4^\dagger$. As we now see, under the action of $\mathfrak{C}$ the block structure (1.129) is preserved and, therefore, it is indeed consistent to put $\Theta_1, \Theta_1^\dagger$ and $\Theta_4, \Theta_4^\dagger$ to zero. In what follows we will assume this gauge choice for the odd part of the coset representative.

Now we will introduce a convenient two-index notation which allows us to naturally treat the dynamical variables of the model as transforming in bi-fundamental representations of $\mathfrak{su}(2)$. To this end, we recall that any SU(2)-matrix $\mathfrak{g}$ obeys the following property:

$$\mathfrak{g}^\dagger = \mathfrak{g}^{-1} = \epsilon\mathfrak{g}^t\epsilon^{-1} \quad \Longrightarrow \quad \mathfrak{g}^* = \epsilon\mathfrak{g}\epsilon^{-1}, \tag{1.134}$$

which provides an equivalence between an irrep of SU(2) and its complex conjugate.

Consider, e.g., matrix Z and multiply it by ϵ from the right. According to equation (1.132), $Z\epsilon$ transforms under $\mathfrak{C}$ as follows:

$$Z\epsilon \to \mathfrak{g}_1 Z \mathfrak{g}_2^{-1}\epsilon = \mathfrak{g}_1 Z\epsilon \mathfrak{g}_2^t. \tag{1.135}$$

If we now associate the index $\alpha = 3, 4$ to the fundamental irrep of $\mathfrak{g}_1$ and the index $\dot\alpha = \dot 3, \dot 4$ to the fundamental irrep of $\mathfrak{g}_2$, then $Z\epsilon$ can be regarded as the matrix with entries $Z^{\alpha\dot\alpha}$

$$Z\epsilon = \begin{pmatrix} Z^{3\dot 3} & Z^{3\dot 4} \\ Z^{4\dot 3} & Z^{4\dot 4} \end{pmatrix}. \tag{1.136}$$

Then formula (1.135) written in components takes the form

$$Z'^{\alpha\dot\alpha} = \mathfrak{g}^{\alpha}_{\beta}\mathfrak{g}^{\dot\alpha}_{\dot\beta} Z^{\beta\dot\beta},$$

which shows that $Z\epsilon$ transforms in the bi-fundamental representation of $\mathfrak{su}(2)$. The matrix Z itself is expressed via the entries of $Z\epsilon$ as

$$Z = \begin{pmatrix} Z^{3\dot 4} & -Z^{3\dot 3} \\ Z^{4\dot 4} & -Z^{4\dot 3} \end{pmatrix}. \tag{1.137}$$

Analogously, we associate the indices $a = 1, 2$ and $\dot a = \dot 1, \dot 2$ with the third and the fourth copies of $\mathfrak{su}(2)$ in equation (1.127), respectively. Then, parametrization (1.129) of the bosonic Lie algebra element $\mathbb{X}$ in terms of the entries $Z^{\alpha\dot\alpha}$ and $Y^{a\dot a}$ reads as follows:

$$\mathbb{X} = \left(\begin{array}{cccc|cccc} 0 & 0 & Z^{3\dot 4} & -Z^{3\dot 3} & 0 & 0 & 0 & 0 \\ 0 & 0 & Z^{4\dot 4} & -Z^{4\dot 3} & 0 & 0 & 0 & 0 \\ -Z^{4\dot 3} & Z^{3\dot 3} & 0 & 0 & 0 & 0 & 0 & 0 \\ -Z^{4\dot 4} & Z^{3\dot 4} & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{i}Y^{1\dot 2} & -\mathrm{i}Y^{1\dot 1} \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{i}Y^{2\dot 2} & -\mathrm{i}Y^{2\dot 1} \\ 0 & 0 & 0 & 0 & -\mathrm{i}Y^{2\dot 1} & \mathrm{i}Y^{1\dot 1} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\mathrm{i}Y^{2\dot 2} & \mathrm{i}Y^{1\dot 2} & 0 & 0 \end{array}\right). \tag{1.138}$$

To obtain this formula, we have replaced the matrices $Z^\dagger$ and $Y^\dagger$ in equation (1.129) via Z and Y by using the reality condition (1.131). In a similar fashion we deduce the following parametrization of the fermionic Lie algebra element χ

$$\chi = \left(\begin{array}{cccc|cccc} 0 & 0 & 0 & 0 & 0 & 0 & \eta^{3\dot 2} & -\eta^{3\dot 1} \\ 0 & 0 & 0 & 0 & 0 & 0 & \eta^{4\dot 2} & -\eta^{4\dot 1} \\ 0 & 0 & 0 & 0 & \theta^\dagger_{1\dot 4} & \theta^\dagger_{2\dot 4} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\theta^\dagger_{1\dot 3} & -\theta^\dagger_{2\dot 3} & 0 & 0 \\ \hline 0 & 0 & \theta^{1\dot 4} & -\theta^{1\dot 3} & 0 & 0 & 0 & 0 \\ 0 & 0 & \theta^{2\dot 4} & -\theta^{2\dot 3} & 0 & 0 & 0 & 0 \\ -\eta^\dagger_{3\dot 2} & -\eta^\dagger_{4\dot 2} & 0 & 0 & 0 & 0 & 0 & 0 \\ \eta^\dagger_{3\dot 1} & \eta^\dagger_{4\dot 1} & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right). \tag{1.139}$$

Here, by definition, $\theta^\dagger_{\alpha\dot a}$ and $\eta^\dagger_{a\dot\alpha}$ are understood as complex conjugate of $\theta^{\alpha\dot a}$ and $\eta^{a\dot\alpha}$, respectively,

$$(\theta^{a\alpha})^* \equiv \theta^\dagger_{a\alpha}, \qquad (\eta^{\alpha a})^* \equiv \eta^\dagger_{\alpha a}. \tag{1.140}$$

In summary, we have shown that the bosonic symmetry algebra $\mathfrak{G}$ which commutes with an element $\Lambda(t,\phi)$ coincides with four copies of $\mathfrak{su}(2)$. The corresponding group acts linearly on the remaining dynamical variables. We choose to parametrize these dynamical variables by fields

$$Z^{\alpha\dot{\alpha}}, \qquad Y^{a\dot{a}}, \qquad \theta^{a\dot{\alpha}}, \qquad \eta^{\alpha\dot{a}},$$

which transform in the bi-fundamental representation of $\mathfrak{su}(2)$.

1.5. Appendix

1.5.1. Embedding coordinates. The bosonic coset element $\mathfrak{g}_\mathfrak{b}$ provides a parametrization of the $\mathrm{AdS}_5\times\mathrm{S}^5$ space in terms of 5 + 5 unconstrained coordinates z^i and y^i. Sometimes it is however more convenient to work with constrained 6 + 6 coordinates which describe the embedding of the AdS_5 and the 5-sphere into $\mathbb{R}^{4,2}$ and $\mathbb{R}^6$, respectively.

The embedding coordinates are defined in the following way. For the 5-sphere we introduce six real coordinates Y_A, $A=1,\ldots,6$ obeying the condition $Y_A^2=1$. These coordinates are related to five unconstrained variables ϕ, y^a as follows:

$$\begin{aligned}
&\mathcal{Y}_1\equiv Y_1+\mathrm{i}Y_2=\frac{y_1+\mathrm{i}y_2}{1+\frac{y^2}{4}}, \qquad \mathcal{Y}_2\equiv Y_3+\mathrm{i}Y_4=\frac{y_3+\mathrm{i}y_4}{1+\frac{y^2}{4}},\\
&\mathcal{Y}_3\equiv Y_5+\mathrm{i}Y_6=\frac{1-\frac{y^2}{4}}{1+\frac{y^2}{4}}\exp(\mathrm{i}\phi).
\end{aligned} \tag{1.141}$$

Here we used the shorthand notation $y^2=y^iy^i$. The metric induced on S^5 from the flat metric of the embedding space is

$$\mathrm{d}Y_A\,\mathrm{d}Y_A=\left(\frac{1-\frac{y^2}{4}}{1+\frac{y^2}{4}}\right)^2\mathrm{d}\phi^2+\frac{\mathrm{d}y_i\,\mathrm{d}y_i}{\left(1+\frac{y^2}{4}\right)^2}. \tag{1.142}$$

Analogously, to describe the five-dimensional AdS space we introduce the embedding coordinates Z_A. These coordinates are constrained to obey $\eta_{AB}Z^AZ^B=-1$ with the metric $\eta_{AB}=(-1,1,1,1,1,-1)$ and are related to t, z^a as

$$\begin{aligned}
&\mathcal{Z}_1\equiv Z_1+\mathrm{i}Z_2=\frac{z_1+\mathrm{i}z_2}{1-\frac{z^2}{4}}, \qquad \mathcal{Z}_2\equiv Z_3+\mathrm{i}Z_4=\frac{z_3+\mathrm{i}z_4}{1-\frac{z^2}{4}},\\
&\mathcal{Z}_3\equiv Z_0+\mathrm{i}Z_5=\frac{1+\frac{z^2}{4}}{1-\frac{z^2}{4}}\exp(\,\mathrm{i}t),
\end{aligned} \tag{1.143}$$

where $z^2=z^iz^i$. For the induced metric one obtains

$$\eta_{AB}\,\mathrm{d}Z^A\,\mathrm{d}Z^B=-\left(\frac{1+\frac{z^2}{4}}{1-\frac{z^2}{4}}\right)^2\mathrm{d}t^2+\frac{\mathrm{d}z_i\,\mathrm{d}z_i}{\left(1-\frac{z^2}{4}\right)^2}. \tag{1.144}$$

In the last formula we assume $-\infty<t<\infty$, which corresponds to considering the universal cover of the AdS space without closed time-like curves. In what follows we do not distinguish between the lower and upper indices for the z- and y-coordinates, that is $z_i\equiv z^i$, $y_i\equiv y^i$. For future convenience we combine the coordinates z^i, y^i into a single vector x^μ with $\mu=1,\ldots,8$ for which $x^i=z^i$, $x^{i+4}=y^i$.

Thus, the metric of the $\mathrm{AdS}_5\times\mathrm{S}^5$ space takes the following diagonal form:

$$\begin{aligned}
\mathrm{d}s^2&=-G_{tt}\,\mathrm{d}t^2+G_{\phi\phi}\,\mathrm{d}\phi^2+G_{zz}\,\mathrm{d}z_i\,\mathrm{d}z_i+G_{yy}\,\mathrm{d}y_i\,\mathrm{d}y_i\\
&=-G_{tt}\,\mathrm{d}t^2+G_{\phi\phi}\,\mathrm{d}\phi^2+G_{\mu\mu}\,\mathrm{d}x^\mu\mathrm{d}x^\mu,
\end{aligned} \tag{1.145}$$

where

$$G_{tt} = \left(\frac{1+\frac{z^2}{4}}{1-\frac{z^2}{4}}\right)^2, \qquad G_{\phi\phi} = \left(\frac{1-\frac{y^2}{4}}{1+\frac{y^2}{4}}\right)^2, \qquad G_{zz} = \frac{1}{\left(1-\frac{z^2}{4}\right)^2}, \qquad G_{yy} = \frac{1}{\left(1+\frac{y^2}{4}\right)^2},$$

and $G_{ii} = G_{zz}, G_{4+i,4+i} = G_{yy}$ for $i = 1, \ldots, 4$.

Having introduced the embedding coordinates, we would like to ask whether there exists a bosonic coset representative $\mathfrak{g}_{\mathfrak{b}}$ such that the bilinear form $\mathrm{str}\big(\mathfrak{g}_{\mathfrak{b}}^{-1}\,\mathrm{d}\mathfrak{g}_{\mathfrak{b}}\big)^2$ would coincide with the metric (1.145). Introduce the following matrices:

$$\mathfrak{g}_{\mathfrak{b}} = \Lambda(t,\phi)\mathfrak{g}(\mathbb{X}), \qquad \mathfrak{g}(\mathbb{X}) = \left(\frac{\mathbb{1}+\mathbb{X}}{\mathbb{1}-\mathbb{X}}\right)^{\frac{1}{2}}, \tag{1.146}$$

where $\mathbb{X}$ is the Lie algebra element (1.122). Substituting here the matrix representation for $\mathbb{X}$, we find the following result:

$$\mathfrak{g}(\mathbb{X}) = \begin{pmatrix} \frac{1}{\sqrt{1-\frac{z^2}{4}}}\left[\mathbb{1}+\frac{1}{2}z^i\gamma^i\right] & 0 \\ 0 & \frac{1}{\sqrt{1+\frac{y^2}{4}}}\left[\mathbb{1}+\frac{\mathrm{i}}{2}y^i\gamma^i\right] \end{pmatrix}. \tag{1.147}$$

One can easily verify that[16] $\mathfrak{g}(\mathbb{X})^\dagger H\mathfrak{g}(\mathbb{X}) = H$, i.e. $\mathfrak{g}_{\mathfrak{b}}$ belongs to the bosonic subgroup of $\mathfrak{su}(2,2|4)$. It depends on t, ϕ and $\mathbb{X}$, i.e. it comprises the coset degrees of freedom corresponding to the $\mathrm{AdS}_5\times\mathrm{S}^5$ space. Thus, we can consider $\mathfrak{g}_{\mathfrak{b}}$ as another embedding of the coset element into $\mathrm{SU}(2,2)\times\mathrm{SU}(4)$, alternative to the exponential map (1.122). Finally, computing the metric $\mathrm{str}(\mathfrak{g}_{\mathfrak{b}}^{-1}\,\mathrm{d}\mathfrak{g}_{\mathfrak{b}})^2$, we see that it reproduces (1.145).

1.5.2. Alternative form of the string Lagrangian. We start with an alternative description of the bosonic coset element and further use it to construct another convenient representation of the string Lagrangian (1.35).

Let $\mathfrak{g}$ be an arbitrary matrix from $\mathrm{SU}(2,2)\times\mathrm{SU}(4)$. Construct the following matrix:

$$\mathrm{G} = \mathfrak{g}\mathcal{K}\mathfrak{g}^t. \tag{1.148}$$

Obviously, G is skew-symmetric: $\mathrm{G}^t = -\mathrm{G}$. It is also pseudo-unitary: $\mathrm{G}^\dagger H\,\mathrm{G} = H$. Let $\mathfrak{h}\in\mathrm{SO}(4,1)\times\mathrm{SO}(5)$. Then $\mathfrak{h}$ leaves the matrix $\mathcal{K}$ invariant: $\mathfrak{h}\mathcal{K}\mathfrak{h}^t = \mathcal{K}$. Therefore, under the right multiplication $\mathfrak{g}\to\mathfrak{g}\mathfrak{h}$ the matrix G remains unchanged

$$\mathfrak{g}\mathcal{K}\mathfrak{g}^t \to \mathfrak{g}\mathfrak{h}\mathcal{K}\mathfrak{h}^t\mathfrak{g}^t = \mathrm{G}.$$

Thus, G depends solely on the coset degrees of freedom comprising the $\mathrm{AdS}_5\times\mathrm{S}^5$ space. This space itself can be thought of as an intersection of (even) pseudo-unitary and skew-symmetric matrices.

Computing now G corresponding to the bosonic element $\mathfrak{g} = \Lambda(t,\phi)\mathfrak{g}(\mathbb{X})$ with $\mathfrak{g}(\mathbb{X})$ given by equation (1.147), we find

$$\mathrm{G} = \Lambda\frac{\mathbb{1}+\mathbb{X}}{\mathbb{1}-\mathbb{X}}\mathcal{K}\Lambda = \left[\Lambda^2\frac{\mathbb{1}+\mathbb{X}^2}{\mathbb{1}-\mathbb{X}^2} - 2\frac{\mathbb{X}}{\mathbb{1}-\mathbb{X}^2}\right]\mathcal{K}, \tag{1.149}$$

where we have used the property $\mathbb{X}\Lambda = \Lambda^{-1}\mathbb{X}$. We see that, opposite to $\mathfrak{g}$, the element G depends on Λ^2 rather than on Λ. Thus, G is a periodic function of σ irrespective of a winding

[16] The map $\mathbb{X}\to\frac{\mathbb{1}+\mathbb{X}}{\mathbb{1}-\mathbb{X}}$ is the Cayley transform which maps a (pseudo-) anti-Hermitian matrix into a (pseudo-) unitary one. In equation (1.146) one can replace the square root by any real function $f(x)$ which admits a power series expansion around $x = 0$. Also, note that $\mathfrak{g}^{-1}(\mathbb{X}) = \mathfrak{g}(-\mathbb{X})$.

sector. Another way to see this is to write G in terms of global embedding coordinates. Representing

$$\mathrm{G} = \begin{pmatrix} \mathrm{G}_{\mathfrak{ads}} & 0 \\ 0 & \mathrm{G}_{\mathfrak{sphere}} \end{pmatrix}, \tag{1.150}$$

we obtain

$$\mathrm{G}_{\mathfrak{ads}} = \begin{pmatrix} 0 & -\mathcal{Z}_3 & \mathcal{Z}_1^* & \mathcal{Z}_2^* \\ \mathcal{Z}_3 & 0 & -\mathcal{Z}_2 & \mathcal{Z}_1 \\ -\mathcal{Z}_1^* & \mathcal{Z}_2 & 0 & -\mathcal{Z}_3^* \\ -\mathcal{Z}_2^* & -\mathcal{Z}_1 & \mathcal{Z}_3^* & 0 \end{pmatrix}, \qquad \mathrm{G}_{\mathfrak{sphere}} = \begin{pmatrix} 0 & -\mathcal{Y}_3 & -\mathrm{i}\mathcal{Y}_1^* & -\mathrm{i}\mathcal{Y}_2^* \\ \mathcal{Y}_3 & 0 & \mathrm{i}\mathcal{Y}_2 & -\mathrm{i}\mathcal{Y}_1 \\ \mathrm{i}\mathcal{Y}_1^* & -\mathrm{i}\mathcal{Y}_2 & 0 & -\mathcal{Y}_3^* \\ \mathrm{i}\mathcal{Y}_2^* & \mathrm{i}\mathcal{Y}_1 & \mathcal{Y}_3^* & 0 \end{pmatrix},$$

where the entries above are written in terms of complex embedding coordinates given by equations (1.141) and (1.143).

We point out that the actual convenience of the embedding coordinates is explained by the fact that, in opposite to x^μ, under the action of the whole bosonic symmetry group they transform linearly. Indeed, if $G \in \mathrm{SU}(2,2)\times\mathrm{SU}(4)$, then

$$G\cdot\mathfrak{g} = \mathfrak{g}'\mathfrak{h}_c \quad\Longrightarrow\quad \mathrm{G}\to\mathrm{G}' = G\cdot\mathrm{G}\cdot G^t,$$

because the compensating element $\mathfrak{h}$ from $\mathrm{SO}(5,1)\times\mathrm{SO}(5)$ decouples from G' as a consequence of definition (1.148).

String Lagrangian. Starting with the coset parametrization (1.119), we write down the corresponding 1-form A

$$A = -\mathfrak{g}^{-1}\,\mathrm{d}\mathfrak{g} = -\mathfrak{g}_{\mathfrak{b}}^{-1}\big(\mathfrak{g}_{\mathfrak{f}}^{-1}\,\mathrm{d}\mathfrak{g}_{\mathfrak{f}}\big)\mathfrak{g}_{\mathfrak{b}} - \mathfrak{g}_{\mathfrak{b}}^{-1}\,\mathrm{d}\mathfrak{g}_{\mathfrak{b}}^{-1}. \tag{1.151}$$

The element $\mathfrak{g}_{\mathfrak{f}}^{-1}\,\mathrm{d}\mathfrak{g}_{\mathfrak{f}}$ takes values in $\mathfrak{su}(2,2|4)$ and it is the sum of even and odd components denoted by B and F, respectively,

$$\mathfrak{g}_{\mathfrak{f}}^{-1}\,\mathrm{d}\mathfrak{g}_{\mathfrak{f}} = \mathrm{B}+\mathrm{F}.$$

Hence, $A = A_{\mathrm{e}} + A_{\mathrm{o}}$, where the even, A_{e}, and odd, A_{o}, components are

$$A_{\mathrm{e}} = -\mathfrak{g}_{\mathfrak{b}}^{-1}\,\mathrm{B}\mathfrak{g}_{\mathfrak{b}} - \mathfrak{g}_{\mathfrak{b}}^{-1}\,\mathrm{d}\mathfrak{g}_{\mathfrak{b}}^{-1}, \qquad A_{\mathrm{o}} = -\mathfrak{g}_{\mathfrak{b}}^{-1}\,\mathrm{F}\mathfrak{g}_{\mathfrak{b}}. \tag{1.152}$$

It is interesting to note that with this choice of the coset parametrization the even component of the flat current is a gauge transformation of the even element B, while the odd one is the adjoint transform of F with the bosonic matrix $\mathfrak{g}_{\mathfrak{b}}$.

As the next step, we compute the $\mathbb{Z}_4$-projections $A^{(k)}$ of the connection (1.152). Straightforward application of formulae (1.24) together with the definition (1.148) gives

$$\begin{aligned} 2A^{(0)} &= A_{\mathrm{e}} - \mathcal{K}A_{\mathrm{e}}^t\mathcal{K}^{-1} = -2\mathfrak{g}_{\mathfrak{b}}^{-1}\,\mathrm{d}\mathfrak{g} - \mathfrak{g}_{\mathfrak{b}}^{-1}(\mathrm{B} - \mathrm{G}\,\mathrm{B}^t\,\mathrm{G} - \mathrm{d}\,\mathrm{G}\,\mathrm{G}^{-1})\mathfrak{g}_{\mathfrak{b}}, \\ 2A^{(2)} &= A_{\mathrm{e}} + \mathcal{K}A_{\mathrm{e}}^t\mathcal{K}^{-1} = -\mathfrak{g}_{\mathfrak{b}}^{-1}(\mathrm{B} + \mathrm{G}\,\mathrm{B}^t\,\mathrm{G} + \mathrm{d}\,\mathrm{G}\,\mathrm{G}^{-1})\mathfrak{g}_{\mathfrak{b}}, \end{aligned} \tag{1.153}$$

for the even components of A, and

$$\begin{aligned} 2A^{(1)} &= A_{\mathrm{o}} + \mathrm{i}\mathcal{K}A_{\mathrm{o}}^{\mathrm{st}}\mathcal{K}^{-1} = -\mathfrak{g}_{\mathfrak{b}}^{-1}(\mathrm{F} + \mathrm{i}\,\mathrm{G}\,\mathrm{F}^{\mathrm{st}}\,\mathrm{G}^{-1})\mathfrak{g}_{\mathfrak{b}}, \\ 2A^{(3)} &= A_{\mathrm{o}} - \mathrm{i}\mathcal{K}A_{\mathrm{o}}^{\mathrm{st}}\mathcal{K}^{-1} = -\mathfrak{g}_{\mathfrak{b}}^{-1}(\mathrm{F} - \mathrm{i}\,\mathrm{G}\,\mathrm{F}^{\mathrm{st}}\,\mathrm{G}^{-1})\mathfrak{g}_{\mathfrak{b}}, \end{aligned} \tag{1.154}$$

for the odd ones. Substituting these projections into the Lagrangian density (1.35), we obtain

$$\begin{aligned} \mathscr{L} = -\frac{g}{8}\,\mathrm{str}\big[&\gamma^{\alpha\beta}(\mathrm{B}_\alpha + \mathrm{G}\,\mathrm{B}_\alpha\,\mathrm{G}^{-1} + \partial_\alpha\,\mathrm{G}\,\mathrm{G}^{-1})(\mathrm{B}_\beta + \mathrm{G}\,\mathrm{B}_\beta\,\mathrm{G}^{-1} + \partial_\beta\,\mathrm{G}\,\mathrm{G}^{-1}) \\ &- 2\mathrm{i}\kappa\epsilon^{\alpha\beta}\,\mathrm{F}_\alpha\,\mathrm{G}\,\mathrm{F}_\beta^{\mathrm{st}}\,\mathrm{G}^{-1}\big]. \end{aligned} \tag{1.155}$$

The nice feature of this Lagrangian is that it only involves the fields which carry the linear representation of the bosonic symmetry algebra.

Finally, the form (1.155) provides a shortcut to reproduce the Polyakov Lagrangian for strings on $AdS_5 \times S^5$, when fermions are switched off. Indeed, putting fermions to zero reduces expression (1.155) to

$$\mathscr{L} = -\frac{g}{8}\gamma^{\alpha\beta}\,\mathrm{str}(\partial_\alpha\,\mathrm{G}\,\mathrm{G}^{-1}\partial_\beta\,\mathrm{G}\,\mathrm{G}^{-1}), \tag{1.156}$$

which is the Lagrangian density for a nonlinear sigma-model with bosonic fields taking value in the $AdS_5 \times S^5$ space described by a group element G.

1.6. Bibliographic remarks

A manifestly supersymmetric covariant flat space superstring action has been found in [53] based on the covariant action for superparticles [54]. This action exhibits κ-symmetry [53] which generalizes the local fermionic symmetries first discovered for massive and massless superparticles [55, 56]. For an introduction to the Green–Schwarz formalism and further references on the covariant quantization issue we refer the reader to the book [9]. Interpretation of the Green–Schwarz string as a coset sigma-model of the Wess–Zumino type has been proposed by Henneaux and Mezincescu [57]. It was shown in [58] that type IIB superstring can be consistently coupled to a generic supergravity background with preservation of κ-symmetry gauge invariance, see also an earlier work [59] on the same subject for the ten-dimensional superstring with $N = 1$ target-space supersymmetry.

The action for type IIB superstring on $AdS_5 \times S^5$ was constructed by Metsaev and Tseytlin [12] along the lines of the Henneaux–Mezincescu approach [57]. Various aspects of this action, alternative formulations and related models have been discussed in [60–62]. In [63] it was found that the Wess–Zumino term entering the sigma model action is d-exact and can be written in the local fashion provided the subgroup H defining the coset space G/H is the invariant locus of a $\mathbb{Z}_4$-automorphism of G. Our exposition of the string sigma model based on the coset space (1.1) follows closely [64].

There is a vast literature on Lie superalgebras. The reader is invited to consult [65–67]. Automorphisms of simple Lie superalgebras have been classified in [68] and we mention the corresponding classification for $\mathfrak{sl}(4|4)$ in section 1.1.2.

Our treatment of κ-symmetry in section 1.2.3 is based on the observation that this symmetry can be understood as the right local action on the coset space supplied with the proper transformation of the two-dimensional world-sheet metric [69]. The reader might also find some similarities with the corresponding discussion in [62].

Concerning the general concept of integrability and conservation laws, we refer the reader to the books [70, 71]. Dynamics of bosonic strings propagating in the $AdS_5 \times S^5$ geometry is described by the corresponding nonlinear sigma model. This model inherits its classical and quantum integrability from the principal chiral model based on the group $SO(4,2) \times SO(6)$. Classically the model is conformally invariant but at the quantum level it develops a mass gap.

Integrability of classical superstring theory on $AdS_5 \times S^5$ has been established for the first time in [8] by exhibiting the zero-curvature representation of the string equations of motion. The corresponding (full and bosonic) Lax pair and the associated conservation laws have been further studied in many papers, see, e.g. [64, 72–78]. The relation between κ-symmetry and integrability was emphasized in [8, 79].

Coset parametrization of the type $\mathfrak{g} = \mathfrak{g}_\mathfrak{f}\mathfrak{g}_\mathfrak{b}$ has been introduced in [80]. Also, the action of the global symmetry algebra on a coset representative was analyzed there. Representation (1.123), suitable for the light-cone gauge fixing, appeared in [14]. The κ-symmetry gauge choice (1.87) was pointed out in [14, 80]. Two-index notation to encode the transformation properties of the world-sheet fields with respect to the linearly realized bosonic subgroup $SU(2)^4$ was introduced in [81]. For the alternative parametrization of the coset space discussed in appendix 2 we refer to works [80] and [82]. The latter paper also contains the alternative form of the string Lagrangian—equation (1.155).

2. Strings in the light-cone gauge

To fix the reparametrization freedom of the string sigma model, in this section we introduce a special one-parameter class of gauges. They are usually called the uniform light-cone gauges. In the light-cone gauge the string sigma model is a two-dimensional field theory defined on a cylinder of circumference P_+ with the light-cone Hamiltonian depending on the string tension g and P_+. It describes a sector of string states, all carrying the same spacetime light-cone momentum P_+. Not all of these states are considered to be physical—a physical state should satisfy the level-matching condition that is its *total world-sheet momentum* must vanish.

Quantization of the light-cone string sigma model simplifies greatly in the so-called decompactification limit where the light-cone momentum tends to infinity, while the string tension is kept fixed. In the decompactification limit the gauge-fixed model is defined on the plane and has massive excitations. Giving up the level-matching condition defines the theory *off-shell*. In the off-shell theory world-sheet excitations (particles) carry non-trivial world-sheet momenta and can scatter among themselves. Their pairwise scattering is encoded into the two-body *world-sheet S-matrix*.

In this section the light-cone model is quantized perturbatively in the large string tension expansion. At the leading order the model is nothing else but a massive relativistic two-dimensional theory with eight bosons and eight fermions. Developing the expansion in powers of $1/g$, one can compute the corresponding perturbative world-sheet S-matrix. We present here the corresponding calculation in the tree-level (Born) approximation. We also study the symmetry algebra of the light-cone model and show that in the off-shell theory it undergoes a central extension; the latter turns out to be crucial for fixing the matrix structure of the exact world-sheet S-matrix.

2.1. Light-cone gauge

In this section we introduce the first-order formalism for the Green–Schwarz superstring in $AdS_5 \times S^5$. Then we impose the uniform light-cone gauge and fix κ-symmetry. The uniform light-cone gauge generalizes the standard phase-space light-cone gauge to a curved background, and it is distinguished from other possible light-cone gauges by the choice of the light-cone coordinates and κ-symmetry fixing. To make the discussion clearer, we start by considering bosonic strings, and then include fermions and fix κ-symmetry.

2.1.1. Bosonic strings in light-cone gauge. We consider strings propagating in a target manifold possessing (at least) two Abelian isometries realized by shifts of the time coordinate of the manifold denoted by t, and a space coordinate denoted by ϕ. If the variable ϕ is an angle then the range of ϕ is chosen to be from 0 to 2π.

To impose a uniform gauge, we also assume that the string sigma-model action is invariant under shifts of t and ϕ, all the other bosonic and fermionic fields being invariant under the shifts. This means that the string action does not have an explicit dependence on t and ϕ and depends only on the derivatives of the fields. An example of such a string action is provided by the Green–Schwarz superstring in $AdS_5 \times S^5$ where the metric can be written in the form, see (1.145)

$$ds^2 = -G_{tt}\,dt^2 + G_{\phi\phi}\,d\phi^2 + G_{\mu\nu}\,dx^\mu\,dx^\nu. \tag{2.1}$$

Here t is the global time coordinate of AdS_5, ϕ is an angle parametrizing the equator of S^5, and x^μ, $\mu = 1, \ldots, 8$, are the remaining 'transversal' coordinates of $AdS_5 \times S^5$.

In this subsection we consider only the bosonic part of a string sigma model action, and assume that the B-field vanishes.

The corresponding part of the string action is of the following form:

$$S = -\frac{g}{2}\int_{-r}^{r} d\sigma\, d\tau\, \gamma^{\alpha\beta}\partial_\alpha X^M \partial_\beta X^N G_{MN}, \tag{2.2}$$

where g is the effective dimensionless string tension, $X^M = \{t, \phi, x^\mu\}$ are string coordinates and G_{MN} is the target-space metric independent of t and ϕ.

The simplest way to impose a uniform light-cone gauge is to introduce momenta canonically conjugate to the coordinates X^M

$$p_M = \frac{\delta S}{\delta \dot{X}^M} = -g\gamma^{0\beta}\partial_\beta X^N G_{MN}, \qquad \dot{X}^M \equiv \partial_0 X^M,$$

and rewrite the string action (2.2) in the first-order form

$$S = \int_{-r}^{r} d\sigma\, d\tau \left(p_M \dot{X}^M + \frac{\gamma^{01}}{\gamma^{00}} C_1 + \frac{1}{2g\gamma^{00}} C_2 \right). \tag{2.3}$$

The reparametrisation invariance of the string action leads to the two Virasoro constraints

$$C_1 = p_M X'^M, \qquad C_2 = G^{MN} p_M p_N + g^2 X'^M X'^N G_{MN}, \qquad X'^M \equiv \partial_1 X^M,$$

which are to be solved after imposing a gauge condition.

The invariance of the string action under the shifts of the time and space coordinates, t and ϕ, of the manifold leads to the existence of two conserved charges

$$E = -\int_{-r}^{r} d\sigma\, p_t, \qquad J = \int_{-r}^{r} d\sigma\, p_\phi. \tag{2.4}$$

It is clear that the charge E is the target-spacetime energy, and J is the U(1) charge of the string equal to the total (angular) momentum of the string in the ϕ-direction.

To impose a uniform gauge we introduce the 'light-cone' coordinates and momenta

$$x_- = \phi - t, \qquad x_+ = (1-a)t + a\phi, \qquad p_- = p_\phi + p_t, \qquad p_+ = (1-a)p_\phi - ap_t,$$
$$t = x_+ - ax_-, \qquad \phi = x_+ + (1-a)x_-, \qquad p_t = (1-a)p_- - p_+, \qquad p_\phi = p_+ + ap_-.$$

Here a is an arbitrary number which parametrizes the most general uniform gauge (up to some trivial rescaling of the light-cone coordinates) such that the light-cone momentum p_- is equal to $p_\phi + p_t$. This choice of gauge is natural in the AdS/CFT context because, as we will see in a moment, in such a uniform gauge the world-sheet Hamiltonian is equal to $E - J$.

Taking into account (2.4), we get the following expressions for the light-cone momenta:

$$P_- = \int_{-r}^{r} d\sigma\, p_- = J - E, \qquad P_+ = \int_{-r}^{r} d\sigma\, p_+ = (1-a)J + aE.$$

In terms of the light-cone coordinates the action (2.3) takes the form

$$S = \int_{-r}^{r} d\sigma\, d\tau \left(p_- \dot{x}_+ + p_+ \dot{x}_- + p_\mu \dot{x}^\mu + \frac{\gamma^{01}}{\gamma^{00}} C_1 + \frac{1}{2g\gamma^{00}} C_2 \right), \tag{2.5}$$

where

$$C_1 = p_+ x'_- + p_- x'_+ + p_\mu x'^\mu. \tag{2.6}$$

The second Virasoro constraint is a quadratic polynomial in p_- which can be cast in the following form:

$$C_2 = \left(a^2 G^{-1}_{\phi\phi} - (a-1)^2 G^{-1}_{tt}\right) p_-^2 + 2\left(a G^{-1}_{\phi\phi} - (a-1) G^{-1}_{tt}\right) p_- p_+ + \left(G^{-1}_{\phi\phi} - G^{-1}_{tt}\right) p_+^2 \\ + g^2((a-1)^2 G_{\phi\phi} - a^2 G_{tt}) x_-'^2 - 2g^2((a-1) G_{\phi\phi} - a G_{tt}) x_-' x_+' \\ + g^2(G_{\phi\phi} - G_{tt}) x_+'^2 + \mathcal{H}_x, \tag{2.7}$$

where $\mathcal{H}_x$ is the part of the constraint which depends only on the transversal fields x^μ and p_μ

$$\mathcal{H}_x = G^{\mu\nu} p_\mu p_\nu + g^2 x'^\mu x'^\nu G_{\mu\nu},$$

and we assume that the target spacetime metric is of -form (2.1).

We then fix the uniform light-cone gauge by imposing the conditions

$$x_+ = \tau + am\sigma, \qquad p_+ = 1. \tag{2.8}$$

The condition $p_+ = 1$ means that the light-cone momentum is distributed uniformly along the string, and this explains the word 'uniform' in the name of the gauge. The integer number m is the winding number which represents the number of times the string winds around the circle parametrized by ϕ. The winding number appears because we consider closed strings and the coordinate ϕ is an angle variable with the range $0 \leqslant \phi \leqslant 2\pi$ and, therefore, it has to satisfy the constraint

$$\phi(r) - \phi(-r) = 2\pi m, \qquad m \in \mathbb{Z}. \tag{2.9}$$

The consistency of this gauge choice also fixes the constant r to be equal to

$$r = \tfrac{1}{2} P_+,$$

which means that the light-cone string sigma model is defined on a cylinder of circumference equal to the total light-cone momentum P_+.

To find the gauge-fixed action, we first solve the Virasoro constraint C_1 for x_-'

$$C_1 = x_-' + amp_- + p_\mu x'^\mu = 0 \qquad \Longrightarrow \qquad x_-' = -amp_- - p_\mu x'^\mu,$$

then we substitute the solution into C_2 and solve the resulting quadratic equation for p_-. Substituting all these solutions into the string action (2.5) and omitting the total derivative $\dot{x}_-^{(0)}$ of the zero mode of x_-, we end up with the gauge-fixed action

$$S = \int_{-r}^{r} \mathrm{d}\sigma\, \mathrm{d}\tau\, (p_\mu \dot{x}^\mu - \mathcal{H}), \tag{2.10}$$

where

$$\mathcal{H} = -p_-(p_\mu, x^\mu, x'^\mu) \tag{2.11}$$

is the density of the world-sheet Hamiltonian which depends only on the physical fields p_μ, x^μ. It is worth noting that $\mathcal{H}$ has no dependence on P_+, and the dependence of the gauge-fixed action and the world-sheet Hamiltonian $H = \int_{-r}^{r} \mathrm{d}\sigma \mathcal{H}$ on P_+ comes only through the integration bounds $\pm r$.

Since we consider closed strings, the transversal fields x^μ are periodic: $x^\mu(r) = x^\mu(-r)$. Therefore, the gauge-fixed action defines a two-dimensional model on a cylinder of circumference $2r = P_+$. In addition, the physical states should also satisfy the level-matching condition

$$\Delta x_- = \int_{-r}^{r} \mathrm{d}\sigma\, x_-' = amH - \int_{-r}^{r} \mathrm{d}\sigma\, p_\mu x'^\mu = 2\pi m, \tag{2.12}$$

that follows by integrating the Virasoro constraint C_1 (2.6) over σ and taking into account that ϕ is an angle variable.

The gauge-fixed action is obviously invariant under the shifts of the world-sheet coordinate σ. This leads to the existence of the conserved charge

$$p_{\rm ws} = -\int_{-r}^{r} d\sigma\, p_\mu x'^\mu, \tag{2.13}$$

which is just the total world-sheet momentum of the string. In what follows we will be mostly interested in the zero-winding number case, $m = 0$ because only in this case the large tension perturbative expansion is well defined. Then the level-matching condition simply states that the total world-sheet momentum vanishes for physical configurations

$$\Delta x_- = p_{\rm ws} = 0, \qquad m = 0. \tag{2.14}$$

It is worth stressing that to quantize the light-cone string sigma model and to also identify its symmetry algebra, one has to consider all states with periodic x^μ and to impose the level-matching condition singling out the physical subspace only at the very end. In a uniform light-cone gauge one has a well-defined model on a cylinder. However, if a string configuration does not satisfy the level-matching condition then its target spacetime image is an open string with end points moving in unison so that Δx_- remains constant. Another peculiarity is related to the fact that the gauge-fixed string sigma models are equivalent for different choices of a uniform gauge, i.e. for different values of a, provided the level-matching condition is satisfied. String configurations which violate the level-matching condition may depend on a. This gauge-dependence makes the problem of quantizing string theory in a uniform gauge very subtle. On the other hand, the requirement that physical states are gauge-independent should impose severe constraints on the structure of the theory. It may also happen that for finite J there is a preferred choice of the parameter a simplifying the exact quantization of the model. In fact, we will see that for finite J the choice $a = 0$ seems to be the most natural one, at least in the AdS/CFT context. For example, only in the $a = 0$ uniform gauge one can study string configurations with an arbitrary winding number in one go.

Let us now consider in more detail bosonic strings in $AdS_5 \times S^5$ where the metric takes the form (1.145). We consider string states with zero-winding number $m = 0$ and impose the uniform light-cone gauge (2.8) $x_+ = \tau$, $p_+ = 1$. Solving the first Virasoro constraint C_1 (2.6) for x'_-, we get

$$x'_- = -p_\mu x'^\mu,$$

while the second constraint (2.7) takes the following form:

$$\begin{aligned} C_2 = \left(a^2 G^{-1}_{\phi\phi} - (a-1)^2 G^{-1}_{tt}\right) p_-^2 + 2\left(a G^{-1}_{\phi\phi} - (a-1) G^{-1}_{tt}\right) p_- + G^{-1}_{\phi\phi} - G^{-1}_{tt} \\ + g^2((a-1)^2 G_{\phi\phi} - a^2 G_{tt}) x_-'^2 + \mathcal{H}_x. \end{aligned} \tag{2.15}$$

There are two solutions of the constraint equation $C_2 = 0$, and one should keep those that leads to a positive definite Hamiltonian density through the relation $\mathcal{H} = -p_-$. A simple computation shows that the solution is given by the following expression:

$$\begin{aligned} \mathcal{H} = \frac{\sqrt{G_{\phi\phi}G_{tt}(1 + ((a-1)^2 G_{\phi\phi} - a^2 G_{tt})\mathcal{H}_x + g^2((a-1)^2 G_{\phi\phi} - a^2 G_{tt})^2 x_-'^2)}}{(a-1)^2 G_{\phi\phi} - a^2 G_{tt}} \\ + \frac{(a-1)G_{\phi\phi} - a G_{tt}}{(a-1)^2 G_{\phi\phi} - a^2 G_{tt}}. \end{aligned} \tag{2.16}$$

The world-sheet light-cone Hamiltonian has a very complicated nonlinear dependence on the physical coordinates and momenta, and it could hardly be used to perform a direct canonical quantization of the model.

The gauge-fixed action corresponding to the world-sheet Hamiltonian[17] can be used to analyze string theory in various limits. One well-known limit is the BMN limit in which one takes $g \to \infty$ and $P_+ \to \infty$, while keeping g/P_+ fixed. In this case it is useful to rescale σ so that the range of σ will be from $-\pi$ to π. The gauge-fixed action then admits a well-defined expansion in powers of $1/g$ (or equivalently $1/P_+$), with the leading part being just a quadratic action for eight massive bosons (and eight fermions for the full model). The action can be easily quantized perturbatively and subsequently used to compute $1/P_+$ corrections to the energy of string states.

Another interesting limit is the decompactification limit where $P_+ \to \infty$ with g kept fixed. In this limit the circumference $2r$ goes to infinity and we get a two-dimensional massive model defined on a plane. Since the gauge-fixed theory is defined on a plane the asymptotic states and S-matrix are well defined. An important observation is that in the limit the light-cone string sigma model admits one- and multi-soliton solutions. The corresponding one-soliton solutions were named giant magnons because they are dual to field theory spin chain magnons and also because generically their size is of order of the radius of S^5. Since for a giant magnon Δx_- is not an integer multiple of 2π, such a soliton configuration does not describe a closed string. We will discuss giant magnons in the following section in detail.

Let us also mention that the world-sheet Hamiltonian in the light-cone gauge is related to the target spacetime energy E and the U(1) charge J as follows:

$$H = \int_{-r}^{r} \mathrm{d}\sigma \mathcal{H} = E - J. \tag{2.17}$$

According to the AdS/CFT correspondence, the spacetime energy E of a string state is identified with the conformal dimension Δ of the dual CFT operator: $E \equiv \Delta$. Since the Hamiltonian H is a function of $P_+ = (1-a)J + aE$, for generic values of a relation (2.17) gives us a non-trivial equation on the energy E. Computing the spectrum of H and solving equation (2.17) would allow one to find conformal dimensions of dual CFT operators.

There are three natural choices of the parameter a. If $a = 0$ we have the temporal gauge $t = \tau$, $P_+ = J$. In this gauge the world-sheet Hamiltonian depends on J only and therefore its spectrum immediately determines the spacetime energy E. If $a = \frac{1}{2}$, we obtain the usual light-cone gauge $x_+ = \frac{1}{2}(t + \phi) = \tau$, $P_+ = \frac{1}{2}(E + J)$. The light-cone gauge appears to drastically simplify perturbative computations in the large tension limit, as we will demonstrate later in this section. Finally, one can also set $a = 1$. In this case, the uniform gauge reduces to $x_+ = \phi = \tau$, $P_+ = E$, where the angle variable ϕ is identified with the world-sheet time τ, and the energy E is distributed uniformly along the string. String theory in $\mathrm{AdS}_5 \times \mathrm{S}^5$ has not been analyzed in this gauge yet.

2.1.2. First-order formalism. To generalize the discussion of the previous subsection to the Green–Schwarz superstring in $\mathrm{AdS}_5 \times \mathrm{S}^5$, one should use the parametrization (1.123) of the coset element that ensures that all fermions are neutral under the U(1) isometries generated by shifts of t and ϕ. Then, to impose the light-cone gauge in the Hamiltonian setting, one should first determine the momenta canonically conjugate to the coordinates t and ϕ (or, equivalently, to the light-cone coordinates $x_\pm$). Because of non-trivial interactions between bosonic and fermionic fields, to find the momenta is not straightforward. A better way to proceed is to

[17] The action is written in the first-order formalism. It is not difficult to see, however, that one can eliminate the momenta from the action by using their equations of motion, and get an action which depends only on x^μ and their first derivatives.

introduce a Lie-algebra valued auxiliary field π, and rewrite the superstring Lagrangian (1.35) in the form

$$\mathscr{L} = -\mathrm{str}\left(\pi A_0^{(2)} + \kappa\frac{g}{2}\epsilon^{\alpha\beta}A_\alpha^{(1)}A_\beta^{(3)} + \frac{\gamma^{01}}{\gamma^{00}}\pi A_1^{(2)} - \frac{1}{2g\gamma^{00}}\left(\pi^2 + g^2\big(A_1^{(2)}\big)^2\right)\right). \tag{2.18}$$

It is easy to see that if one solves the equations of motion for π and substitutes the solution back into (2.18) one obtains (1.35). The last two terms in (2.18) yield the Virasoro constraints

$$C_1 = \mathrm{str}\pi A_1^{(2)} = 0, \tag{2.19}$$

$$C_2 = \mathrm{str}\big(\pi^2 + g^2\big(A_1^{(2)}\big)^2\big) = 0, \tag{2.20}$$

which are to be solved after imposing the light-cone gauge and fixing the κ-symmetry.

It is clear that without loss of generality we can assume that π belongs to the subspace $M^{(2)}$ of $\mathfrak{su}(2,2|4)$, as the other components in the $\mathbb{Z}_4$ grading decouple. It therefore admits the following decomposition:

$$\pi = \pi^{(2)} = \frac{\mathrm{i}}{2}\pi_+\Sigma_+ + \frac{\mathrm{i}}{4}\pi_-\Sigma_- + \frac{1}{2}\pi_\mu\Sigma_\mu + \pi_{\mathbb{1}}\,\mathrm{i}\mathbb{1}_8. \tag{2.21}$$

where Σ's are 8×8 matrices defined as follows:

$$\Sigma_+ = \begin{pmatrix}\Sigma & 0\\ 0 & \Sigma\end{pmatrix}, \qquad \Sigma_- = \begin{pmatrix}-\Sigma & 0\\ 0 & \Sigma\end{pmatrix}, \qquad \Sigma_k = \begin{pmatrix}\gamma_k & 0\\ 0 & 0\end{pmatrix}, \qquad \Sigma_{4+k} = \begin{pmatrix}0 & 0\\ 0 & \mathrm{i}\gamma_k\end{pmatrix}. \tag{2.22}$$

Since $A_\alpha^{(2)}$ belongs to the superalgebra $\mathfrak{su}(2,2|4)$, $\mathrm{str}A_\alpha^{(2)} = 0$, and the quantity $\pi_{\mathbb{1}}$ does not contribute to the Lagrangian.

It is worth stressing that the fields $\pi_\pm$ do not coincide with the momenta $p_\pm$ canonically conjugate to $x_\mp$ but they can be expressed in terms of $p_\pm$. Before doing this, we impose the κ-symmetry gauge, which will dramatically simplify our further treatment.

2.1.3. Kappa-symmetry gauge fixing. As was discussed in the previous section, the key property of the Green–Schwarz action is its invariance under the fermionic κ-symmetry that halves the number of fermionic degrees of freedom. A κ-symmetry gauge should be compatible with the bosonic gauge imposed, and the analysis of the κ-symmetry transformations (1.72) for the Green–Schwarz superstring action (2.18) performed in subsection 1.2.3 revealed that for the uniform light-cone gauge the κ-symmetry could be fixed by choosing the fermion field χ (1.118) to be of the form (1.139). It is not difficult to check that the gauge-fixed fermion field χ satisfies the following important relations:

$$\Sigma_+\chi = -\chi\Sigma_+, \qquad \Sigma_-\chi = \chi\Sigma_-. \tag{2.23}$$

In fact these relations may be considered as the defining ones for the κ-symmetry gauge we have chosen and can be used instead of specifying the explicit form of χ. Taking into account that $\mathfrak{g}^{-1}(\chi) = \mathfrak{g}(-\chi)$ and these identities, one can then easily show that

$$\begin{aligned}\mathfrak{g}^{-1}(\chi)\Sigma_+ = \Sigma_+\mathfrak{g}(\chi) \quad &\Rightarrow \quad \mathfrak{g}^{-1}(\chi)\Sigma_+\mathfrak{g}(\chi) = \Sigma_+\mathfrak{g}(\chi)^2,\\ \mathfrak{g}^{-1}(\chi)\Sigma_- = \Sigma_-\mathfrak{g}^{-1}(\chi) \quad &\Rightarrow \quad \mathfrak{g}^{-1}(\chi)\Sigma_-\mathfrak{g}(\chi) = \Sigma_-.\end{aligned}$$

The perturbative expansion of the light-cone Lagrangian in powers of the fields simplifies if one uses $\mathfrak{g}(x) \equiv \mathfrak{g}(\mathbb{X})$ of the form (1.146), and the matrix $g(\chi)$ of the form

$$g(\chi) = \chi + \sqrt{1+\chi^2}. \tag{2.24}$$

The standard exponential parametrization (1.118) can be obtained from (2.24) by means of the following fermion redefinition $\chi \to \sinh\chi$.

Now it is straightforward to use the coset parametrization (1.123) to compute the current (1.33)

$$A = A_{\text{even}} + A_{\text{odd}},$$

where

$$\begin{aligned} A_{\text{even}} = -\mathfrak{g}^{-1}(x)\left[\frac{\mathrm{i}}{2}\left(\mathrm{d}x_+ + \left(\frac{1}{2}-a\right)\mathrm{d}x_-\right)\Sigma_+(1+2\chi^2) + \frac{\mathrm{i}}{4}\,\mathrm{d}x_-\Sigma_-\right]\mathfrak{g}(x) \\ -\mathfrak{g}^{-1}(x)\big[\sqrt{1+\chi^2}\,\mathrm{d}\sqrt{1+\chi^2} - \chi\,\mathrm{d}\chi + \mathrm{d}\mathfrak{g}(x)\mathfrak{g}^{-1}(x)\big]\mathfrak{g}(x), \end{aligned} \tag{2.25}$$

$$\begin{aligned} A_{\text{odd}} = -\mathfrak{g}^{-1}(x)\Bigg[\,\mathrm{i}\left(\mathrm{d}x_+ + \left(\frac{1}{2}-a\right)\mathrm{d}x_-\right)\Sigma_+\chi\sqrt{1+\chi^2} \\ +\sqrt{1+\chi^2}\,\mathrm{d}\chi - \chi\,\mathrm{d}\sqrt{1+\chi^2}\Bigg]\mathfrak{g}(x). \end{aligned} \tag{2.26}$$

These formulae clearly demonstrate that the currents acquire the simplest form if the parameter a of the uniform light-cone gauge is equal to $1/2$. For $a = 1/2$ the odd part of the current A does not depend on the light-cone coordinate x_-! This explains the drastic simplifications that occur for the $a = 1/2$ gauge in comparison to the general uniform gauge. For $a = 1/2$ and in the gauge $x_+ = \tau$ the odd part of A depends on the derivatives of the fermion χ only. In what follows we restrict our discussion of the fermionic part of the light-cone Green–Schwarz action to the simplest case $a = 1/2$.

2.1.4. Light-cone gauge fixing. Now we can use the formulae established above to express $\pi_\pm$ in terms of $p_\pm$. To this end, omitting the Virasoro constraints, we can rewrite the Lagrangian (2.18) as follows:

$$\mathscr{L} = p_+\dot{x}_- + \mathbf{p}_-\dot{x}_+ - \mathrm{str}\left(\pi A^{\perp}_{\text{even}} + \kappa\frac{g}{2}\epsilon^{\alpha\beta}A^{(1)}_\alpha A^{(3)}_\beta\right), \tag{2.27}$$

where

$$A^{\perp}_{\text{even}} = -\mathfrak{g}^{-1}(x)\big[\sqrt{1+\chi^2}\partial_\tau\sqrt{1+\chi^2} - \chi\partial_\tau\chi + \partial_\tau\mathfrak{g}(x)\mathfrak{g}^{-1}(x)\big]\mathfrak{g}(x),$$

and the momentum p_+, canonically conjugate to x_-, is shown to be equal to

$$p_+ = \frac{\mathrm{i}}{4}\,\mathrm{str}(\pi\Sigma_-\mathfrak{g}(x)^2) = G_+\pi_+ - \frac{1}{2}G_-\pi_-, \qquad G_\pm = \frac{1}{2}\left(G_{tt}^{\frac{1}{2}} \pm G_{\phi\phi}^{\frac{1}{2}}\right). \tag{2.28}$$

The variable $\mathbf{p}_-$ is not equal to the momentum p_- canonically conjugate to x_+. It differs from p_- by a contribution coming from the Wess–Zumino term in (2.27) , and is defined as follows:

$$\mathbf{p}_- = \frac{\mathrm{i}}{2}\,\mathrm{str}\left(\pi\Sigma_+\mathfrak{g}(x)(1+2\chi^2)\mathfrak{g}(x)\right). \tag{2.29}$$

Now having identified the light-cone momentum p_+, we can impose the uniform light-cone gauge (2.8) with $a = 1/2$

$$x_+ = \tau + \tfrac{1}{2}m\sigma, \qquad p_+ = 1. \tag{2.30}$$

Let us stress again that the density $\mathcal{H}$ of the world-sheet light-cone Hamiltonian is equal to $-p_-$ but not to $-\mathbf{p}_-$.

It is also important to recall that to impose the light-cone gauge we had to make all the fermions of the string sigma model neutral with respect to the two U(1) isometry groups generated by the shifts of t and ϕ. As a result, in the light-cone gauge the fermions are periodic in the even winding sector and they are anti-periodic in the odd winding sector.

In what follows we will be interested in the decompactification and large string tension limits, and, therefore, we set $m = 0$.

2.1.5. Gauge-fixed Lagrangian. Now we are ready to find the light-cone gauge-fixed Lagrangian. This is a multi-step procedure. First, we solve equation (2.28) determining p_+ for $\pi_+ = \pi_+(p_+, \pi_-)$ and set $p_+ = 1$ in the solution. Second, we solve the Virasoro constraint C_1 of equation (2.19) to find x'_-. Finally, we determine π_- from the second Virasoro constraint C_2 equation (2.7). Substituting all the solutions into the Lagrangian of equation (2.27), we end up with the total gauge-fixed Lagrangian. The explicit derivation is rather involved and we refer the reader to the original literature for details, see section 2.6.

The upshot is a Lagrangian which can be written in the standard form as the difference of a kinetic term and the Hamiltonian density

$$\mathscr{L}_{gf} = \mathscr{L}_{\rm kin} - \mathcal{H}. \tag{2.31}$$

The kinetic term $\mathscr{L}_{\rm kin}$ depends on the time derivatives of the physical fields, and determines the Poisson structure of the theory. It can be cast in the form

$$\begin{aligned}\mathscr{L}_{\rm kin} = p_\mu \dot{x}_\mu - \frac{\rm i}{2}\,{\rm str}(\Sigma_+\chi\partial_\tau\chi) + \frac{1}{2}\mathfrak{g}_\nu\pi_\mu{\rm str}([\Sigma_\nu, \Sigma_\mu]B_\tau)\\ - {\rm i}\kappa\frac{g}{2}\left(G_+^2 - G_-^2\right){\rm str}\left(F_\tau\mathcal{K}F_\sigma^{\rm st}\mathcal{K}\right) + {\rm i}\kappa\frac{g}{2}G_\mu G_\nu{\rm str}\left(\Sigma_\nu F_\tau\Sigma_\mu\mathcal{K}F_\sigma^{\rm st}\mathcal{K}\right),\end{aligned} \tag{2.32}$$

where we use the following definitions:

$$\mathfrak{g}(x) = \mathfrak{g}_+ I_8 + \mathfrak{g}_-\Upsilon + \mathfrak{g}_\mu\Sigma_\mu, \qquad \mathfrak{g}(x)^2 = G_+ I_8 + G_-\Upsilon + G_\mu\Sigma_\mu,$$

and the functions B_α and F_α refer to the even and odd components of $\mathfrak{g}^{-1}(\chi)\partial_\alpha\mathfrak{g}(\chi)$

$$\begin{aligned}&\mathfrak{g}^{-1}(\chi)\partial_\alpha\mathfrak{g}(\chi) = B_\alpha + F_\alpha,\\ &B_\alpha = -\tfrac{1}{2}\chi\partial_\alpha\chi + \tfrac{1}{2}\partial_\alpha\chi\chi + \tfrac{1}{2}\sqrt{1+\chi^2}\partial_\alpha\sqrt{1+\chi^2} - \tfrac{1}{2}\partial_\alpha\sqrt{1+\chi^2}\sqrt{1+\chi^2},\\ &F_\alpha = \sqrt{1+\chi^2}\partial_\alpha\chi - \chi\partial_\alpha\sqrt{1+\chi^2}.\end{aligned} \tag{2.33}$$

As one can see, the kinetic term is highly non-trivial and leads to a complicated Poisson structure. To quantize the theory perturbatively, e.g. in the large string tension limit, one would need to redefine the fields so that the kinetic term acquires the conventional form

$$\mathscr{L}_{\rm kin} \to p_\mu\dot{x}_\mu - \frac{\rm i}{2}\,{\rm str}\,(\Sigma_+\chi\partial_\tau\chi)\,, \tag{2.34}$$

and, therefore, the redefined fields would satisfy the canonical commutation relations. This will be done in the following section up to the quartic order in the fields.

The density $\mathcal{H}$ of the Hamiltonian is given by the sum of $-\mathbf{p}_-$ and the Wess–Zumino term

$$\mathcal{H} = -\mathbf{p}_- + \mathcal{H}_{WZ}, \tag{2.35}$$

where

$$\begin{aligned}\mathcal{H}_{WZ} = -\kappa\frac{g}{2}\left(G_+^2 - G_-^2\right){\rm str}\big(\Sigma_+\chi\sqrt{1+\chi^2}\mathcal{K}F_\sigma^{\rm st}\mathcal{K}\big)\\ - \kappa\frac{g}{2}G_\mu G_\nu{\rm str}\big(\Sigma_+\Sigma_\nu\chi\sqrt{1+\chi^2}\Sigma_\mu\mathcal{K}F_\sigma^{\rm st}\mathcal{K}\big).\end{aligned}$$

Let us stress that, in this way, we find the gauge-fixed Lagrangian as an *exact* function of the string tension g. The corresponding light-cone gauge-fixed action is written in the standard form

$$S_{gf} = \int_{-r}^{r} {\rm d}\sigma\,{\rm d}\tau\,\mathscr{L}_{gf}, \qquad r = P_+/2, \tag{2.36}$$

and its dependence on the total light-cone momentum P_+ comes only through the integration bounds, as it was in the bosonic case discussed in the previous subsection. Then it is straightforward to take the decompactification limit and get a two-dimensional model on the plane. This will be discussed in detail in the following section.

The gauge-fixed Lagrangian and Hamiltonian are obviously invariant under the transformations generated by the $SU(2)^4$ bosonic subgroup of the PSU(2,2|4) supergroup discussed in subsection 1.4.2 because the subgroup commutes with the $\mathfrak{u}(1)$-isometries corresponding to shifts of t and ϕ, and, therefore, preserves the light-cone and κ-symmetry gauge-fixing conditions.

Finally, the physical states should satisfy the level-matching condition which is obtained by integrating the Virasoro constraint C_1 (2.19) over σ

$$\Delta x_- = \int_{-r}^{r} \mathrm{d}\sigma\, x'_- = -\int_{-r}^{r} \mathrm{d}\sigma \left(p_\mu x'_\mu - \frac{\mathrm{i}}{2}\,\mathrm{str}(\Sigma_+ \chi \chi') + \frac{1}{2} g_\nu \pi_\mu \mathrm{str}([\Sigma_\nu, \Sigma_\mu] B_\sigma) \right). \tag{2.37}$$

The right-hand side of the equation is equal to the world-sheet momentum carried by the string, and, since we consider the zero-winding number sector, it must vanish for closed strings

$$\Delta x_- = p_{\mathrm{ws}} = 0.$$

2.2. Decompactification limit

In this section we discuss properties of the light-cone string theory in the decompactification limit where the total light-cone momentum P_+ goes to infinity, and one gets a massive two-dimensional model defined on the plane. The resulting model possesses multi-soliton solutions, and we construct the simplest one-soliton solution and find its dispersion relation. Then, we study the structure of the model in the large tension perturbative expansion, perform its perturbative quantization, identify closed sectors, and construct a perturbative world-sheet S-matrix which satisfies the classical Yang–Baxter equation.

2.2.1. From cylinder to plane. The light-cone string sigma model Hamiltonian constructed in the previous section describes a highly nonlinear two-dimensional model defined on a cylinder, and it is obviously too complicated to be quantized and solved exactly by using canonical methods. A better way to address the spectral problem is to first consider the states carrying very large light-cone momentum P_+, and then to take into account the finite P_+ effects.

As was shown in the previous section, the light-cone string sigma model action is of the following form:

$$S = \int_{-r}^{r} \mathrm{d}\sigma\, \mathrm{d}\tau \mathscr{L},$$

where $r = P_+/2$, and the Lagrangian density $\mathscr{L}$ depends on the string tension g, but it has no dependence on the light-cone momentum P_+. The light-cone model is defined on a cylinder, and this is reflected in the periodic boundary conditions imposed on the bosonic and fermionic fields entering the Lagrangian. A physical configuration corresponding to a closed string must satisfy the level-matching condition which is equivalent to the vanishing of its world-sheet momentum.

The specific dependence of the action on the light-cone momentum P_+ allows one to define the decompactification or infinite light-cone momentum limit. In this limit the circumference of the cylinder tends to infinity, while the string tension is kept fixed, and one is left with a non-trivial interacting model defined on the plane. The periodic boundary conditions turn into the vanishing ones because one is interested in string states with finite world-sheet energy.

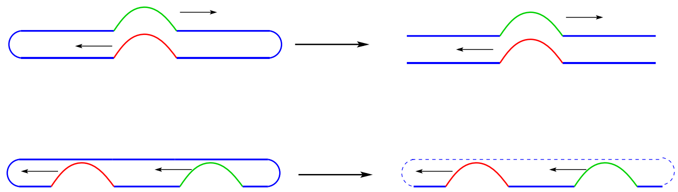

Figure 2. Solitonic excitations of a closed string in the decompactification limit.

Since $H = E - J$ is finite, and $P_+ = (1-a)J + aE \to \infty$, the charge J also goes to infinity in the decompactification limit.

The resulting model appears to be non-Lorentz-invariant but it has massive spectrum, and, therefore, the notion of particles and their scattering matrix is well-defined. Moreover, this model is expected to be integrable at the quantum level, and hence multi-body interactions should factorize into a sequence of two-body events. Thus, in the decompactification limit the problem of solving the theory reduces to three steps: first identify the asymptotic spectrum, i.e. elementary excitations and their bound states, second find the dispersion relations for all the particles and, finally, determine the two-body S-matrices. It is worth stressing, however, that in order to be able to consider particles with arbitrary momenta, one should go off-shell, i.e. to give up the level-matching condition and allow for unphysical configurations that do not correspond to closed strings. As a result, some quantities, e.g. the world-sheet S-matrix, acquire a mild gauge dependence.

At leading order in the large tension expansion the light-cone model describes eight free bosons and eight free fermions of equal mass. The quadratic action, in fact, coincides with the light-cone action for superstrings in the plane-wave background and, for this reason, the large tension expansion is sometimes called the near plane wave one. This expansion is rather peculiar because one can easily perform the perturbative quantization of the light-cone model and perturbatively compute the world-sheet S-matrix that describes scattering of massive bosons and fermions. In order to determine the exact S-matrix, one has to use more sophisticated methods to be developed in section 3.

An interesting feature of the light-cone string sigma model in the decompactification limit is that it admits (multi-)soliton solutions, see figure 2. Below we discuss the simplest one-soliton solution called the giant magnon.

2.2.2. Giant magnon. To construct classical solutions of the light-cone string sigma model it is sufficient to consider only its bosonic part. In general a solution may involve several fields from both the AdS and S^5 parts of the background geometry. Simplest solutions would obviously depend only on one field, and one can show that a solution with a finite energy which can therefore be called a soliton exists only if one takes a field from the 5-sphere.

The corresponding part of the gauge-fixed string action is obtained from the Hamiltonian (2.16) by setting to zero all the fields but one, say y_1, from the S^5 part of the action. One can easily check that it is a consistent reduction of the light-cone model. Then, it is convenient to make the following change of variables:

$$z = \frac{y_1}{1 + \frac{y_1^2}{4}}.$$

In the conformal gauge the corresponding reduction of the string sigma model is to strings moving in the $\mathbb{R}\times S^2$ part of the $AdS_5\times S^5$ background. In terms of the angle coordinate ϕ and z the metric of S^2 takes the form

$$ds^2_{S^2}=\frac{dz^2}{1-z^2}+(1-z^2)\,d\phi^2.$$

The coordinate z is related to the angle θ as $z=\cos\theta$. The values $z=\pm1$ correspond to the north and south poles of the sphere, and at $z=0$ the angle ϕ parametrizes the equator.

The light-cone Hamiltonian depends on the string tension, and it is convenient to rescale the world-sheet coordinate σ as $\sigma\to g\sigma$. Then, the light-cone action takes the following form:

$$S=g\int_{-\infty}^{\infty}d\sigma\,d\tau\,(p_z\dot{z}-\mathcal{H})\,, \tag{2.38}$$

where the density of the gauge-fixed Hamiltonian is a function of the coordinate z and its canonically conjugate momentum p_z, but it has no dependence on the string tension g. Explicit expressions for the Hamiltonian and other quantities computed in this subsection can be found in appendix 2.5.1 where we also present their forms for the three simplest cases $a=0,1/2,1$.

To find soliton solutions of the gauge-fixed string theory, it is convenient to go to the Lagrangian description by eliminating the momentum p_z. Solving the equation of motion for p_z that follows from the action (2.38), we determine the momentum as a function of $\dot{z}$ and z. Then substituting the solution into (2.38), we obtain the action in the Lagrangian form: $S=S(z,z',\dot{z})$. The explicit form of the action is given in appendix 2.5.1, and it is of the Nambu–Goto form. We will see in a moment that this leads to the existence of finite-energy singular solitons.

To find a one-soliton solution, we make the most general ansatz describing a wave propagating along the string

$$z=z(\sigma-v\tau), \tag{2.39}$$

where v is the velocity of the soliton. Substituting the ansatz into the action (2.137) from appendix 2.5.1 , we derive the Lagrangian, $L_{\rm red}=L_{\rm red}(z,z')$, of a reduced model which defines a one-particle system if we regard σ as a time variable. The σ-evolution of this system can be easily determined by introducing the 'momentum' conjugate to z with respect to 'time' σ

$$\pi_z=\frac{\partial L_{\rm red}}{\partial z'},$$

and computing the reduced Hamiltonian

$$H_{\rm red}=\pi_z z'-L_{\rm red}.$$

The reduced Hamiltonian is a conserved quantity with respect to σ. Since the coordinate z of the soliton satisfies vanishing boundary conditions, $z(\pm\infty)=z'(\pm\infty)=0$, we conclude that

$$H_{\rm red}=0.$$

Solving this equation with respect to z', we get the following basic equation:

$$z'^2=\left(\frac{1-z^2}{1-(1-a)z^2}\right)^2\frac{z^2}{1-v^2-z^2}, \tag{2.40}$$

which can be easily integrated in terms of elementary functions.

It is not difficult to see that a solution with finite energy exists if the following inequalities hold:

$$0 \leqslant a \leqslant 1, \qquad 0 \leqslant |v| \leqslant 1. \tag{2.41}$$

Then, assuming for definiteness that $z \geqslant 0$, the corresponding solution of equation (2.40) lies between 0 and $z_{\max} = \sqrt{1-v^2}$. One can easily see from equation (2.40) that in the range of parameters (2.41) the shape of the soliton is similar for any values of a and v. The allowed values of z are $0 \leqslant z \leqslant z_{\max}$, and z' vanishes at $z = 0$, and goes to infinity at $z = z_{\max}$.

The corresponding solution is, as we see, not smooth at $z = z_{\max}$. The energy of this soliton is however finite. To compute the energy, we need to evaluate $\mathcal{H}/|z'|$ on the solution

$$\frac{\mathcal{H}}{|z'|} = \frac{z}{\sqrt{z_{\max}^2 - z^2}}.$$

Then the soliton energy is given by the following integral:

$$E - J = g\int_{-\infty}^{\infty} \mathrm{d}\sigma \mathcal{H} = 2g\int_0^{z_{\max}} \mathrm{d}z \frac{\mathcal{H}}{|z'|} = 2g\sqrt{1-v^2}. \tag{2.42}$$

Finally, to find the dispersion relation we also need to compute the world-sheet momentum (2.13)

$$p_{\mathrm{ws}} = -\int_{-\infty}^{\infty} \mathrm{d}\sigma\, p_z z' = 2\int_0^{z_{\max}} \mathrm{d}z |p_z|, \tag{2.43}$$

where we have assumed that $v > 0$, and took into account that for the soliton we consider the expression $-p_z z'$ is positive. The following explicit formula for the momentum p_z canonically conjugate to z can be easily found by using equations (2.136), (2.39) and (2.40):

$$p_z = \frac{vz}{(1-z^2)\sqrt{z_{\max}^2 - z^2}}. \tag{2.44}$$

Computing the world-sheet momentum

$$p_{\mathrm{ws}} = 2\arccos v,$$

and expressing v in terms of p_{ws}, we derive the giant magnon dispersion relation

$$E - J = 2g\left|\sin\frac{p_{\mathrm{ws}}}{2}\right|. \tag{2.45}$$

The dispersion relation explicitly shows that the light-cone model is not Lorentz-invariant. It appears to be independent of the gauge parameter a. Note also the appearance of trigonometric functions which are usually associated with a lattice structure, while here the dispersion relation was obtained for a continuum model. The dispersion relation was derived in classical theory, i.e. in the limit of large string tension g and finite world-sheet momentum p_{ws}. In the quantum theory it gets modified, and we will discuss the exact dispersion relation in the following section.

Let us finally mention that in the case of a one-soliton solution the world-sheet momentum (2.43) is just equal to the canonical momentum carried by the center of mass of the soliton. To see that we just need to plug the ansatz (2.39) into the string action (2.38), and integrate over σ. Then we obtain the following action for a point particle:

$$S = g\int \mathrm{d}\tau\, (p_{\mathrm{ws}} v - \mathcal{H}),$$

that explicitly shows that p_{ws} is the soliton momentum.

2.2.3. Large string tension expansion. In this subsection we discuss the large string tension expansion and perturbative quantization of the light-cone gauge-fixed action (2.36) in the decompactification limit. To develop the expansion, we first note that the string tension g in the gauge-fixed Lagrangian (2.31) is always accompanied by a σ-derivative of a field. Thus, rescaling[18] the coordinate σ as $\sigma \to g\sigma$ removes the g-dependence from the Lagrangian, and the light-cone action takes the form

$$S_{gf} = g\int \mathrm{d}\sigma\, \mathrm{d}\tau \mathscr{L}_{gf}, \tag{2.46}$$

where $\mathscr{L}_{gf}$ is given by (2.31) with $g = 1$. Finally, one rescales all the fields appearing in (2.46) as follows:

$$x_\mu \to x_\mu/\sqrt{g}, \qquad p_\mu \to p_\mu/\sqrt{g}, \qquad \chi \to \chi/\sqrt{g}, \tag{2.47}$$

and expands the action (2.46) in powers of $1/g$

$$S_{gf} = \int \mathrm{d}\sigma\, \mathrm{d}\tau \left(\mathscr{L}_2 + \frac{1}{g}\mathscr{L}_4 + \frac{1}{g^2}\mathscr{L}_6 + \cdots\right), \tag{2.48}$$

where $\mathscr{L}_2$ is quadratic in the fields, $\mathscr{L}_4$ is quartic, and so on.

It is worth mentioning that the rescaling (2.47) implies the following rescaling of the world-sheet momentum of a state:

$$p_{\mathrm{ws}} = -\int \mathrm{d}\sigma\, (p_\mu x'_\mu + \cdots) = \frac{1}{g}p,$$

where p is the rescaled world-sheet momentum given by the same formula $p = -\int \mathrm{d}\sigma\,(p_\mu x'_\mu + \cdots)$ in terms of the rescaled coordinates and momenta. It is kept fixed in the large tension expansion and, therefore, one considers states with very small world-sheet momenta p_{ws} of order $1/g$.

In principle it is straightforward to expand the light-cone Lagrangian (2.31) and find the quadratic and quartic Lagrangians. The quadratic Lagrangian appears to be of the following simple form:

$$\mathscr{L}_2 = p_\mu \dot{x}_\mu - \frac{\mathrm{i}}{2}\,\mathrm{str}(\Sigma_+\chi\dot{\chi}) - \mathcal{H}_2, \tag{2.49}$$

where the first two terms with time derivatives define the standard Poisson structure for the bosons and fermions, and $\mathcal{H}_2$ is the density of the quadratic Hamiltonian

$$\mathcal{H}_2 = \frac{1}{2}p_\mu^2 + \frac{1}{2}x_\mu^2 + \frac{1}{2}x'^2_\mu - \frac{\kappa}{2}\,\mathrm{str}(\Sigma_+\chi\mathcal{K}\chi'^{\mathrm{st}}\mathcal{K}) + \frac{1}{2}\,\mathrm{str}\chi^2. \tag{2.50}$$

The quadratic Lagrangian obviously describes a Lorentz-invariant theory of eight massive bosons and eight massive fermions with masses equal to unity. It can be easily canonically quantized as we describe in the following subsection.

The quartic Lagrangian obtained just by expanding (2.31), however, has two unpleasant properties. First of all, it contains terms depending on the time derivatives of the fields which come from the interacting part of the kinetic Lagrangian (2.32). These terms modify the Poisson structure and make quantizing the model more complicated. One should remove these terms by redefining the fields.

To find the necessary field redefinition, we note that the kinetic Lagrangian (2.32) can be written in the following form:

$$\mathscr{L}_{\mathrm{kin}} = p_\mu \dot{x}_\mu - \frac{\mathrm{i}}{2}\,\mathrm{str}\,(\Sigma_+\chi\dot{\chi}) + \frac{\mathrm{i}}{g}\,\mathrm{str}\,(\Sigma_+\Phi(p, x, \chi)\dot{\chi}), \tag{2.51}$$

[18] Performing the rescaling with finite P_+ changes the integration bounds in (2.36) as $r \to P_+/2g$.

where Φ is a function of at least cubic order in physical fields. It is then clear that the last term can be removed by the following redefinition of χ:

$$\chi \to \chi + \frac{1}{g}\Phi(p, x, \chi). \tag{2.52}$$

This redefinition casts the kinetic term (2.51) into the form (up to a total derivative)

$$\begin{aligned}\mathscr{L}_{\rm kin} = p_\mu \dot{x}_\mu - \frac{\rm i}{2}\,{\rm str}\,(\Sigma_+\chi\dot{\chi}) + \frac{\rm i}{g}\,{\rm str}\left(\Sigma_+\left(\Phi(p,x,\chi+\frac{1}{g}\Phi) - \Phi(p,x,\chi)\right)\dot{\chi}\right)\\ + \frac{\rm i}{2g^2}\,{\rm str}(\Sigma_+\Phi(p,x,\chi)\dot{\Phi}(p,x,\chi)).\end{aligned} \tag{2.53}$$

Since Φ is at least of cubic order in the fields, the terms on the second line of (2.53) are at least of sixth order. These terms can be also removed by a similar field redefinition. However, this time one would need to redefine not only the fermions but also the bosonic coordinates x_μ and p_μ. For our purposes here it is sufficient to perform only the simplest redefinition (2.52), and just drop the terms on the second line of (2.53). This reduces the kinetic term to the canonical quadratic form which enters the quadratic Lagrangian (2.49). Since the redefinition removes all the time derivative terms from the quartic Lagrangian, the latter becomes equal up to the minus sign to the quartic Hamiltonian: $\mathscr{L}_4 = -\mathcal{H}_4$.

It is also necessary to mention an important and nice property of the redefinition (2.52). One can check that up to sixth order in fields, formula (2.37) for x'_- takes the form

$$x'_- = -\frac{1}{g}\left(p_\mu x'_\mu - \frac{\rm i}{2}\,{\rm str}(\Sigma_+\chi\chi') + \partial_\sigma f(p,x,\chi)\right), \tag{2.54}$$

where $f(p, x, \chi)$ is a function of the momenta and coordinates of at least fourth-order in the fields. Thus, we see that integrating (2.54) over σ, we get the usual 'flat space' form of the level-matching condition and world-sheet momentum

$$\Delta x_- = \int_{-\infty}^{\infty} {\rm d}\sigma\, x'_- = p_{\rm ws} = \frac{p}{g} = -\frac{1}{g}\int_{-\infty}^{\infty} {\rm d}\sigma\left(p_\mu x'_\mu - \frac{\rm i}{2}\,{\rm str}(\Sigma_+\chi\chi')\right). \tag{2.55}$$

Let us stress again that even though for physical states the total world-sheet momentum must vanish, to define asymptotic states and the scattering matrix we should consider states with arbitrary world-sheet momenta.

The second unpleasant property of the quartic Lagrangian (and Hamiltonian) is that it contains bosonic terms of the form p^2x^2 which do not depend on the space derivatives. These terms, however, can be removed by a proper canonical transformation. The final form of the quartic Hamiltonian is

$$\begin{aligned}\mathcal{H}_4 = \frac{1}{4}\Big[&2(y'^2z^2 - z'^2y^2 + z'^2z^2 - y'^2y^2)\\ &- {\rm str}\left(\frac{1}{2}\chi\chi'\chi\chi' + \frac{1}{2}\chi^2\chi'^2 + \frac{1}{4}[\chi,\chi']\mathcal{K}[\chi,\chi']^t\mathcal{K} + \chi\mathcal{K}\chi'^{\rm st}\mathcal{K}\chi\mathcal{K}\chi'^{\rm st}\mathcal{K}\right)\\ &+ {\rm str}\left((z^2-y^2)\chi'\chi' + \frac{1}{2}x'_\mu x_\nu[\Sigma_\mu,\Sigma_\nu][\chi,\chi'] - 2x_\mu x_\nu\Sigma_\mu\chi'\Sigma_\nu\chi'\right)\\ &+ \frac{{\rm i}\kappa}{4}x_\nu p_\mu {\rm str}([\Sigma_\nu,\Sigma_\mu][\mathcal{K}\chi^{\rm st}\mathcal{K},\chi]')\Big].\end{aligned} \tag{2.56}$$

The computation of the quartic Hamiltonian is rather involved, and we refer the reader to the original literature for details. The quadratic and the quartic Hamiltonians can also be written in terms of the bosonic and fermionic matrices $\mathbb{X}$ and χ, see (1.138) and (1.139), as follows:

$$\mathcal{H}_2 = \frac{1}{2}\,{\rm str}(\pi^\perp\pi^\perp + \mathbb{X}\mathbb{X} + \mathbb{X}'\mathbb{X}' - \kappa\Sigma_+\chi\mathcal{K}\chi'^{\rm st}\mathcal{K} + \chi\chi), \tag{2.57}$$

$$
\begin{aligned}
\mathcal{H}_4 = {} & \frac{1}{2}\,\mathrm{str}\Upsilon\mathbb{X}\mathbb{X}\,\mathrm{str}\mathbb{X}'\mathbb{X}' + \frac{1}{4}\,\mathrm{str}\Upsilon\mathbb{X}\mathbb{X}\,\mathrm{str}\chi'\chi' \\
& - \mathrm{str}\left(\frac{1}{2}[\mathbb{X},\mathbb{X}'][\chi,\chi'] + 2\mathbb{X}\chi'\mathbb{X}\chi' - \frac{\mathrm{i}\kappa}{4}[\mathbb{X},\pi^{\perp}][\mathcal{K}\chi^{\mathrm{st}}\mathcal{K},\chi]'\right) \\
& - \mathrm{str}\left(\frac{1}{8}\chi\chi'\chi\chi' + \frac{1}{8}\chi^2\chi'^2 + \frac{1}{16}[\chi,\chi']\mathcal{K}[\chi^t,\chi''] \mathcal{K} + \frac{1}{4}\chi\mathcal{K}\chi'^{\mathrm{st}}\mathcal{K}\chi\mathcal{K}\chi'^{\mathrm{st}}\mathcal{K}\right),
\end{aligned} \tag{2.58}
$$

where $\Upsilon = \mathrm{diag}(\mathbb{1}_4, -\mathbb{1}_4)$, and the momentum $\pi^{\perp} = \frac{1}{2}p_\mu\Sigma_\mu$ has the following form in terms of two-index fields:

$$
\pi^{\perp} = \frac{1}{2}\left(\begin{array}{cccc|cccc}
0 & 0 & -P_{4\dot3} & -P_{4\dot4} & 0 & 0 & 0 & 0 \\
0 & 0 & P_{3\dot3} & P_{3\dot4} & 0 & 0 & 0 & 0 \\
P_{3\dot4} & P_{4\dot4} & 0 & 0 & 0 & 0 & 0 & 0 \\
-P_{3\dot3} & -P_{4\dot3} & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & -\,\mathrm{i}P_{2\dot1} & -\,\mathrm{i}P_{2\dot2} \\
0 & 0 & 0 & 0 & 0 & 0 & \mathrm{i}P_{1\dot1} & \mathrm{i}P_{1\dot2} \\
0 & 0 & 0 & 0 & \mathrm{i}P_{1\dot2} & \mathrm{i}P_{2\dot2} & 0 & 0 \\
0 & 0 & 0 & 0 & -\,\mathrm{i}P_{1\dot1} & -\,\mathrm{i}P_{2\dot1} & 0 & 0
\end{array}\right). \tag{2.59}
$$

The momenta $P_{a\dot a}$ and $P_{\alpha\dot\alpha}$ are canonically conjugate to $Y^{a\dot a}$ and $Z^{\alpha\dot\alpha}$, and $\pi^{\perp}$ satisfies the relation $\mathrm{str}\pi^{\perp}\dot{\mathbb{X}} = p_\mu\dot x^\mu = P_{a\dot a}\dot Y^{a\dot a} + P_{\alpha\dot\alpha}\dot Z^{\alpha\dot\alpha}$. This form also makes the invariance of the Hamiltonians under the transformations generated by the $\mathrm{SU}(2)^4$ subgroup of PSU(2,2|4) manifest.

Summarizing the discussion in this subsection, we conclude that by means of proper field redefinitions at each order of the large g expansion the light-cone gauge-fixed Lagrangian (2.31) can be brought to the following canonical form:

$$
\mathscr{L}_{gf} = \mathrm{str}\left(\pi^{\perp}\dot{\mathbb{X}} - \frac{\mathrm{i}}{2}\Sigma_+\chi\dot\chi\right) - \mathcal{H}_2 - \frac{1}{g}\mathcal{H}_4 - \frac{1}{g^2}\mathcal{H}_6 - \cdots, \tag{2.60}
$$

where the interaction part does not contain terms with the time derivatives, and also terms which do not depend on the space derivatives. Perturbative quantization of the model can be performed in the canonical way by using the quadratic part of the Lagrangian which describes eight massive bosons and eight massive fermions. The quartic Hamiltonian can be then used to compute the tree-level two-particle world-sheet scattering matrix.

2.2.4. Quantization. We now turn to the perturbative quantization of the light-cone $\mathrm{AdS}_5\times\mathrm{S}^5$ superstring in the large tension expansion. We start with rewriting the quadratic Lagrangian density in terms of the two-index fields, see equations (1.138), (1.139) and (2.59).

$$
\mathscr{L}_2 = P_{a\dot a}\dot Y^{a\dot a} + P_{\alpha\dot\alpha}\dot Z^{\alpha\dot\alpha} + \mathrm{i}\eta^{\dagger}_{\alpha\dot a}\dot\eta^{\alpha\dot a} + \mathrm{i}\theta^{\dagger}_{a\dot\alpha}\dot\theta^{a\dot\alpha} - \mathcal{H}_2, \tag{2.61}
$$

where the density of the quadratic Hamiltonian is given by

$$
\begin{aligned}
\mathcal{H}_2 = {} & \frac{1}{4}P_{a\dot a}P^{a\dot a} + Y_{a\dot a}Y^{a\dot a} + Y'_{a\dot a}Y'^{a\dot a} + \frac{1}{4}P_{\alpha\dot\alpha}P^{\alpha\dot\alpha} + Z_{\alpha\dot\alpha}Z^{\alpha\dot\alpha} + Z'_{\alpha\dot\alpha}Z'^{\alpha\dot\alpha} \\
& + \eta^{\dagger}_{\alpha\dot a}\eta^{\alpha\dot a} + \frac{\kappa}{2}\eta^{\alpha\dot a}\eta'_{\alpha\dot a} - \frac{\kappa}{2}\eta^{\dagger\alpha\dot a}\eta'^{\dagger}_{\alpha\dot a} + \theta^{\dagger}_{a\dot\alpha}\theta^{a\dot\alpha} + \frac{\kappa}{2}\theta^{a\dot\alpha}\theta'_{a\dot\alpha} - \frac{\kappa}{2}\theta^{\dagger a\dot\alpha}\theta'^{\dagger}_{a\dot\alpha}.
\end{aligned} \tag{2.62}
$$

Here $\theta^{\dagger}_{a\dot\alpha}$ and $\eta^{\dagger}_{\alpha\dot a}$ are complex conjugate of $\theta^{a\dot\alpha}$ and $\eta^{\alpha\dot a}$, respectively, and we lower and raise the indices by using the ϵ-tensor

$$
Y_{a\dot a} = \epsilon_{ab}\epsilon_{\dot a\dot b}Y^{b\dot b}, \qquad P^{a\dot a} = \epsilon^{ab}\epsilon^{\dot a\dot b}P_{b\dot b}, \qquad \eta_{\alpha\dot a} = \epsilon_{\alpha\beta}\epsilon_{\dot a\dot b}\eta^{\beta\dot b}, \qquad \eta^{\dagger\alpha\dot a} = \epsilon^{\alpha\beta}\epsilon^{\dot a\dot b}\eta^{\dagger}_{\beta\dot b}, \tag{2.63}
$$

and similar formulae for $Z_{\alpha\dot{\alpha}}$, $P^{\alpha\dot{\alpha}}$, $\theta_{a\dot{\alpha}}$, $\theta^{\dagger a\dot{\alpha}}$. The reality condition for these bosonic and fermionic fields then takes the following form:

$$(Y^{a\dot{a}})^\dagger = Y_{a\dot{a}}, \qquad (P_{a\dot{a}})^\dagger = P^{a\dot{a}}, \qquad (\eta_{\alpha\dot{a}})^\dagger = \eta^{\dagger\alpha\dot{a}}.$$

The canonical equal-time (anti)commutation relations for the fields can be now easily read off from (2.61)

$$[Y^{a\dot{a}}(\sigma,\tau), P_{b\dot{b}}(\sigma',\tau)] = \mathrm{i}\delta^a_b\delta^{\dot{a}}_{\dot{b}}\delta(\sigma-\sigma'), \qquad [Z^{\alpha\dot{\alpha}}(\sigma,\tau), P_{\beta\dot{\beta}}(\sigma',\tau)] = \mathrm{i}\delta^\alpha_\beta\delta^{\dot{\alpha}}_{\dot{\beta}}\delta(\sigma-\sigma'),$$
$$\left\{\theta^{a\dot{\alpha}}(\sigma,\tau), \theta^\dagger_{b\dot{\beta}}(\sigma',\tau)\right\} = \delta^a_b\delta^{\dot{\alpha}}_{\dot{\beta}}\delta(\sigma-\sigma'), \qquad \left\{\eta^{\alpha\dot{a}}(\sigma,\tau), \eta^\dagger_{\beta\dot{b}}(\sigma',\tau)\right\} = \delta^\alpha_\beta\delta^{\dot{a}}_{\dot{b}}\delta(\sigma-\sigma'),$$

and we just need to establish a mode decomposition of the bosonic and fermionic fields which renders the quadratic Lagrangian (2.61) in a diagonal form.

We set $\kappa = 1$ for definiteness, and choose the following mode decompositions for the bosonic fields:

$$\begin{aligned}
Y^{a\dot{a}}(\sigma,\tau) &= \frac{1}{\sqrt{2\pi}}\int \mathrm{d}p \frac{1}{2\sqrt{\omega_p}}\big(\mathrm{e}^{\mathrm{i}p\sigma}a^{a\dot{a}}(p,\tau) + \mathrm{e}^{-\mathrm{i}p\sigma}\epsilon^{ab}\epsilon^{\dot{a}\dot{b}}a^\dagger_{b\dot{b}}(p,\tau)\big),\\
P_{a\dot{a}}(\sigma,\tau) &= \frac{1}{\sqrt{2\pi}}\int \mathrm{d}p\, \mathrm{i}\sqrt{\omega_p}\big(\mathrm{e}^{-\mathrm{i}p\sigma}a^\dagger_{a\dot{a}}(p,\tau) - \mathrm{e}^{\mathrm{i}p\sigma}\epsilon_{ab}\epsilon_{\dot{a}\dot{b}}a^{b\dot{b}}(p,\tau)\big),\\
Z^{\alpha\dot{\alpha}}(\sigma,\tau) &= \frac{1}{\sqrt{2\pi}}\int \mathrm{d}p \frac{1}{2\sqrt{\omega_p}}\big(\mathrm{e}^{\mathrm{i}p\sigma}a^{\alpha\dot{\alpha}}(p,\tau) + \mathrm{e}^{-\mathrm{i}p\sigma}\epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}a^\dagger_{\beta\dot{\beta}}(p,\tau)\big),\\
P_{\alpha\dot{\alpha}}(\sigma,\tau) &= \frac{1}{\sqrt{2\pi}}\int \mathrm{d}p\, \mathrm{i}\sqrt{\omega_p}\big(\mathrm{e}^{-\mathrm{i}p\sigma}a^\dagger_{\alpha\dot{\alpha}}(p,\tau) - \mathrm{e}^{\mathrm{i}p\sigma}\epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}}a^{\beta\dot{\beta}}(p,\tau)\big),
\end{aligned} \tag{2.64}$$

and similarly for fermionic ones

$$\begin{aligned}
\theta^{a\dot{\alpha}}(\sigma,\tau) &= \frac{\mathrm{e}^{-\mathrm{i}\pi/4}}{\sqrt{2\pi}}\int \frac{\mathrm{d}p}{\sqrt{\omega_p}}\big(\mathrm{e}^{\mathrm{i}p\sigma}f_p a^{a\dot{\alpha}}(p,\tau) + \mathrm{e}^{-\mathrm{i}p\sigma}h_p\epsilon^{ab}\epsilon^{\dot{\alpha}\dot{\beta}}a^\dagger_{b\dot{\beta}}(p,\tau)\big),\\
\eta^{\alpha\dot{a}}(\sigma,\tau) &= \frac{\mathrm{e}^{-\mathrm{i}\pi/4}}{\sqrt{2\pi}}\int \frac{\mathrm{d}p}{\sqrt{\omega_p}}\big(\mathrm{e}^{\mathrm{i}p\sigma}f_p a^{\alpha\dot{a}}(p,\tau) + \mathrm{e}^{-\mathrm{i}p\sigma}h_p\epsilon^{\alpha\beta}\epsilon^{\dot{a}\dot{b}}a^\dagger_{\beta\dot{b}}(p,\tau)\big).
\end{aligned} \tag{2.65}$$

Here the creation $a^\dagger_{M\dot{M}}$ and annihilation $a^{M\dot{M}}$ operators are conjugate to each other: $(a^{M\dot{M}})^\dagger = a^\dagger_{M\dot{M}}$, where $M = 1,\ldots,4$ and $\dot{M} = \dot{1},\ldots,\dot{4}$; the frequency is $\omega_p = \sqrt{1+p^2}$, and the quantities

$$f_p = \sqrt{\frac{\omega_p+1}{2}}, \qquad h_p = \frac{p}{2f_p}, \qquad f_p^2 - h_p^2 = 1, \qquad f_p^2 + h_p^2 = \omega_p,$$

play the role of the fermion wavfunctions. In what follows we always use capital Latin letters $M, N, \ldots$ and $\dot{M}, \dot{N}, \ldots$ to denote superindices $M = (a|\alpha)$, and $\dot{M} = (\dot{a}|\dot{\alpha})$, where the lower-case Latin indices are even and the Greek indices are odd. Thus, the grading of $M, \dot{M}$ is defined to be $\epsilon_a = \epsilon_{\dot{a}} = 0$ and $\epsilon_\alpha = \epsilon_{\dot{\alpha}} = 1$.

For the sake of simplicity, we will not explicitly show the time dependence in all the operators everywhere where it cannot lead to any confusion. Then, omitting total derivative terms, the quadratic Lagrangian indeed takes the diagonal form

$$L_2 = \int \mathrm{d}\sigma \mathscr{L}_2 = \int \mathrm{d}p \sum_{M,\dot{M}} \big(\, \mathrm{i}a^\dagger_{M\dot{M}}(p)\dot{a}^{M\dot{M}}(p) - \omega_p a^\dagger_{M\dot{M}}(p)a^{M\dot{M}}(p)\big),$$

which shows explicitly that the creation and annihilation operators satisfy the canonical equal-time (anti-)commutation relations

$$\left[a^{M\dot{M}}(p,\tau), a^{\dagger}_{N\dot{N}}(p',\tau)\right\} = \delta^M_N \delta^{\dot{M}}_{\dot{N}} \delta(p-p'), \tag{2.66}$$

where we take the commutator for bosons, and the anti-commutator for fermions.

The quadratic Hamiltonian is, therefore, of the standard harmonic oscillator form

$$\mathbb{H}_2 = \int \mathrm{d}p \sum_{M,\dot{M}} \omega_p a^{\dagger}_{M\dot{M}}(p) a^{M\dot{M}}(p), \tag{2.67}$$

and its generic Q-particle state can now be created by acting with creation operators on the vacuum

$$|\Psi\rangle = a^{\dagger}_{M_1\dot{M}_1}(p_1) a^{\dagger}_{M_2\dot{M}_2}(p_2) \cdots a^{\dagger}_{M_Q\dot{M}_Q}(p_Q)|0\rangle, \tag{2.68}$$

where we may assume that the momenta are ordered as follows:

$$p_1 > p_2 > \cdots > p_{Q-1} > p_Q.$$

The energy of this state is obviously equal to

$$\mathbb{H}_2|\Psi\rangle = E|\Psi\rangle, \qquad E = \sum_i \omega_{p_i}.$$

This state is also an eigenvector of the world-sheet momentum operator which takes the following form:

$$\begin{aligned}\mathbb{P} \equiv p_{\mathrm{ws}} &= -\frac{1}{g}\int \mathrm{d}\sigma\left(P_{a\dot{a}} Y'^{a\dot{a}} + P_{\alpha\dot{\alpha}} Z'^{\alpha\dot{\alpha}} + i\theta^{\dagger}_{\alpha\dot{a}}\theta'^{\alpha\dot{a}} + \mathrm{i}\eta^{\dagger}_{a\dot{\alpha}}\eta'^{a\dot{\alpha}}\right)\\ &= \frac{1}{g}\int \mathrm{d}p \sum_{M,\dot{M}} p a^{\dagger}_{M\dot{M}}(p) a^{M\dot{M}}(p).\end{aligned} \tag{2.69}$$

A physical string state must also satisfy the level-matching condition implying that its world-sheet momentum vanishes

$$\mathbb{P}|\Psi\rangle = 0 \Rightarrow \sum_i p_i = 0.$$

Nevertheless, to understand the general properties of the scattering matrix we would need to consider states with arbitrary momenta.

The tree-level two-particle scattering matrix is determined by the quartic Hamiltonian $\mathbb{H}_4$ that we take to be normal ordered with respect to these bosonic and fermionic oscillator modes. Its expression in terms of the two-index fields is given in appendix 2.5.2.

2.2.5. Closed sectors. It is clear that there are 16 one-particle states of different flavors, and, therefore, the two-particle scattering matrix is a $(16\times16)\times(16\times16)$ matrix. The S-matrix is not diagonal, and in the scattering process particles can exchange their momenta and flavors. The model is believed to be integrable, and the multi-particle scattering matrix can be expressed through a product of the two-particle ones. There are, however, groups of particles of definite flavors which can scatter only among themselves. They are said to form closed sectors.

The simplest way to identify closed sectors is to use the fact that all the 16 particles are charged under the bosonic $\mathfrak{su}(2)\oplus\mathfrak{su}(2)\oplus\mathfrak{su}(2)\oplus\mathfrak{su}(2)$ subalgebra of the symmetry algebra of the light-cone model, and the total charges carried by the scattering states are preserved in the scattering process. Let us recall that two $\mathfrak{su}(2)$'s belong to $\mathfrak{su}(4)\subset\mathfrak{psu}(2,2|4)$, and act on the undotted and dotted lower-case Latin indices $a, b, \dot{a}, \dot{b}, \ldots$ which take the values 1, 2 and $\dot{1}, \dot{2}$, and that the other two $\mathfrak{su}(2)$'s belong to $\mathfrak{su}(2,2)\subset\mathfrak{psu}(2,2|4)$, and act on the undotted and dotted Greek indices $\alpha, \beta, \dot{\alpha}, \dot{\beta}, \ldots$ which take the values 3, 4 and $\dot{3}, \dot{4}$. Thus, the bosonic fields with all Latin indices come from the 5-sphere, and those with all Greek indices come from the AdS part of the string sigma model. Below we describe some closed sectors.

$\mathfrak{su}(2)$ *sector.* The $\mathfrak{su}(2)$ sector is a rank-one sector which consists of bosonic particles of type $a^\dagger_{1\dot{1}}$ originating from the 5-sphere of $AdS_5 \times S^5$, and a generic Q-particle state from the sector is of the form

$$|\Psi_{\mathfrak{su}(2)}\rangle = a^\dagger_{1\dot{1}}(p_1)a^\dagger_{1\dot{1}}(p_2)\cdots a^\dagger_{1\dot{1}}(p_Q)|0\rangle. \tag{2.70}$$

These states can obviously scatter only among themselves because they carry the maximum charges $Q/2, Q/2$ with respect to $\mathfrak{su}(2)\oplus\mathfrak{su}(2) \subset \mathfrak{su}(4)$. The $\mathfrak{su}(2)\oplus\mathfrak{su}(2)$ algebra is isomorphic to the $\mathfrak{so}(4)$ that rotates the four coordinates y_i from S^5, see section 1.5.1 for detail, and a $a^\dagger_{1\dot{1}}$ particle carries charge 1 with respect to the $\mathfrak{o}(2) \sim \mathfrak{u}(1)$ which rotates the y_1y_2-plane, and charge 0 with respect to the $\mathfrak{o}(2) \sim \mathfrak{u}(1)$ which rotates the y_3y_4-plane.

Field theory operators dual to the states (2.70) with vanishing total world-sheet momentum can be easily identified. To this end we assume that the light-cone momentum $P_+ = \frac{1}{2}(E+J)$ is very large but not infinite. Recall that J is the charge associated with the U(1) generating shifts of the angle ϕ of S^5. Then, the charge J is also large, and it is assigned to the light-cone vacuum and *no* creation and annihilation operator carries charges under this U(1). Thus, the states (2.70) are the lightest states which only carry the two charges J and Q, and they should be dual to the $\mathcal{N}=4$ SYM operators of the form

$$O_{\mathfrak{su}(2)} = \mathrm{tr}(Z^J X^Q + \text{permutations}), \tag{2.71}$$

where Z and X are the two complex gauge theory scalars which carry one unit of the charges J and Q, respectively. Note that there is an $\mathfrak{su}(2)$ algebra which rotates the two complex scalars Z, X, and this explains why the sector is called the $\mathfrak{su}(2)$ sector. It is clear that the particles of type $a^\dagger_{2\dot{2}}$ form another closed $\mathfrak{su}(2)$ sector.

$\mathfrak{sl}(2)$ *sector.* The $\mathfrak{sl}(2)$ sector is a rank-one sector consisting of bosonic particles of type $a^\dagger_{3\dot{3}}$ from the AdS part of $AdS_5 \times S^5$, and a generic $\mathfrak{sl}(2)$ sector Q-particle state is

$$|\Psi_{\mathfrak{sl}(2)}\rangle = a^\dagger_{3\dot{3}}(p_1)a^\dagger_{3\dot{3}}(p_2)\cdots a^\dagger_{3\dot{3}}(p_Q)|0\rangle. \tag{2.72}$$

These states scatter only among themselves because they carry the maximum charges $Q/2, Q/2$ with respect to $\mathfrak{su}(2)\oplus\mathfrak{su}(2) \subset \mathfrak{su}(2,2)$. The $\mathfrak{su}(2)\oplus\mathfrak{su}(2) \sim \mathfrak{so}(4)$ rotates the four coordinates z_i from AdS $_5$, and a $a^\dagger_{3\dot{3}}$ particle carries charges 1 and 0 with respect to the two $\mathfrak{o}(2) \sim \mathfrak{u}(1)$'s which rotate the z_1z_2- and z_3z_4-planes, respectively.

Thus, the states (2.72) are the lightest states which only carry the two charges J and Q, and they are dual to the $\mathcal{N}=4$ SYM operators of the form

$$O_{\mathfrak{sl}(2)} = \mathrm{tr}(D_-^Q Z^J + \text{permutations}), \tag{2.73}$$

where D_- is the covariant derivative in a light-cone direction carrying unit charge under the $\mathfrak{u}(1)$ subalgebra of $\mathfrak{su}(2,2)$ that in the string picture corresponds to the $\mathfrak{o}(2)$ which rotates the z_1z_2-plane. The particles of type $a^\dagger_{4\dot{4}}$ obviously form another closed $\mathfrak{sl}(2)$ sector.

$\mathfrak{su}(1|1)$ *sector.* The $\mathfrak{su}(1|1)$ sector is a rank-one sector consisting of fermionic particles of type $a^\dagger_{3\dot{1}}$, and a generic $\mathfrak{su}(1|1)$ sector Q-particle state is

$$|\Psi_{\mathfrak{su}(1|1)}\rangle = a^\dagger_{3\dot{1}}(p_1)a^\dagger_{3\dot{1}}(p_2)\cdots a^\dagger_{3\dot{1}}(p_Q)|0\rangle. \tag{2.74}$$

These states scatter only among themselves, and are dual to the $\mathcal{N}=4$ SYM operators of the form

$$O_{\mathfrak{su}(1|1)} = \mathrm{tr}(Z^{J-\frac{Q}{2}}\Psi^Q + \text{permutations}). \tag{2.75}$$

The fermion Ψ is the highest weight component of the gaugino from the vector multiplet of the gauge theory. The gaugino Ψ_α belongs to the vector multiplet, it is neutral under $\mathfrak{su}(3)$

which rotates the three gauge theory complex scalars among themselves, and it carries the same charge 1/2 under any of the three U(1) subgroups of SU(4). Note also that there is the second equivalent $\mathfrak{su}(1|1)$ sector consisting of fermionic particles of type $a^\dagger_{1\dot{3}}$.

$\mathfrak{su}(1|2)$ *sector.* The $\mathfrak{su}(1|2)$ sector can be considered as the union of the $\mathfrak{su}(2)$ and $\mathfrak{su}(1|1)$ sectors, because it consists of particles of types $a^\dagger_{1\dot{1}}$ and $a^\dagger_{3\dot{1}}$, and a generic $\mathfrak{su}(1|2)$ sector Q-particle state is

$$|\Psi_{\mathfrak{su}(1|2)}\rangle = a^\dagger_{3\dot{1}}(p_1)a^\dagger_{3\dot{1}}(p_2)\cdots a^\dagger_{3\dot{1}}(p_M)a^\dagger_{1\dot{1}}(k_1)a^\dagger_{1\dot{1}}(k_2)\cdots a^\dagger_{1\dot{1}}(k_K)|0\rangle. \quad (2.76)$$

Counting the charges carried by these states shows that they scatter only among themselves, and the number of bosons and fermions is unchanged in the scattering process.

These states obviously are dual to the $\mathcal{N} = 4$ SYM operators of the form

$$O_{\mathfrak{su}(1|2)} = \mathrm{tr}(Z^{J-\frac{M}{2}}\Psi^M X^K + \text{permutations}), \quad (2.77)$$

because, as was discussed above, the gauge theory fields X and Ψ correspond to the creation operators $a^\dagger_{1\dot{1}}$ and $a^\dagger_{3\dot{1}}$, respectively.

$\mathfrak{su}(2|3)$ *sector.* The $\mathfrak{su}(2|3)$ sector is the largest closed sector, and it is an extension of the $\mathfrak{su}(1|2)$ sector. It involves two bosonic particles of types $a^\dagger_{1\dot{1}}$ and $a^\dagger_{2\dot{1}}$, and two fermionic particles of types $a^\dagger_{3\dot{1}}$ and $a^\dagger_{4\dot{1}}$. A generic Q-particle state in the $\mathfrak{su}(2|3)$ sector is

$$a^\dagger_{3\dot{1}}(p_1)\cdots a^\dagger_{3\dot{1}}(p_{M_+})a^\dagger_{4\dot{1}}(\bar{p}_1)\cdots a^\dagger_{4\dot{1}}(\bar{p}_{M_-})a^\dagger_{1\dot{1}}(k_1)\cdots a^\dagger_{1\dot{1}}(k_{J_1})a^\dagger_{2\dot{1}}(\bar{k}_1)\cdots a^\dagger_{2\dot{1}}(\bar{k}_{J_2})|0\rangle.$$

We see that the left $\mathfrak{su}(2|2)$ subalgebra of the symmetry algebra $\mathfrak{su}(2|2)\oplus\mathfrak{su}(2|2)$ acts on the states of the sector.

The $\mathfrak{su}(2|3)$ sector exhibits the following new feature. One can easily check that the operators $a^\dagger_{1\dot{1}}a^\dagger_{2\dot{1}}$ and $a^\dagger_{3\dot{1}}a^\dagger_{4\dot{1}}$ have the same charges, and, therefore, the scattering of two bosons can result into two fermions. Thus, the number of bosons and fermions is not preserved in the scattering process involving particles from this sector.

These states can be shown to be dual to the $\mathcal{N} = 4$ SYM operators of the form

$$O_{\mathfrak{su}(2|3)} = \mathrm{tr}\big(Z^{J-\frac{M_+}{2}-\frac{M_-}{2}}X^{J_1}Y^{J_2}\Psi_+^{M_+}\Psi_-^{M_-} + \text{permutations}\big), \quad (2.78)$$

where Ψ_+ is the highest weight component of the gaugino Ψ_α from the vector multiplet that was denoted as Ψ previously, and Ψ_- is the lowest weight component.

2.3. Perturbative world-sheet S-matrix

2.3.1. Generalities. In scattering theory the S-matrix is a unitary operator, which we denote by $\mathbb{S}$, mapping free particle *out*-states to free particle *in*-states in the Heisenberg picture. Both *in*- and *out*-states belong to the same Hilbert space of the model, and are eigenvectors of the full Hamiltonian $\mathbb{H}$ with the same eigenvalue E

$$\mathbb{H}|p_1, p_2, \ldots, p_n\rangle^{(in/out)}_{i_1,\ldots,i_n} = E|p_1, p_2, \ldots, p_n\rangle^{(in/out)}_{i_1,\ldots,i_n}. \quad (2.79)$$

Here $i_1, \ldots, i_n$ are flavor indices used to account for different kinds of particles in the model, and p_k is the momentum carried by the particle with the flavor i_k either at $t = -\infty$ for *in*-states or at $t = \infty$ for *out*-states. The eigenvalue E is given by

$$E = \sum_{k=1}^{n} \omega^{(i_k)}_{p_k}, \quad (2.80)$$

where $\omega_p^{(i)}$ is the energy (the dispersion relation) of a particle of type i with the momentum p. Recall that in relativistic theory the dispersion relation is of the form $\omega_p = \sqrt{m^2+p^2}$, where m is the mass of the particle which may depend on coupling constants of the model and may receive quantum corrections; momentum p can take any real value. In a lattice discretization of a relativistic model the dispersion relation appears in the form $\omega_p = \sqrt{m^2+\frac{4}{\ell^2}\sin^2\frac{p}{2}}$, where ℓ is a lattice step and p changes from $-\pi$ to π. As we will see in the following chapter, the exact dispersion relation for particles of the light-cone string theory in the decompactification limit is $\omega_p = \sqrt{1+4g^2\sin^2\frac{p}{2}}$ where g is the string tension, and therefore the quantum light-cone string sigma model can be regarded as a lattice model with the lattice step $\ell = 1/g$. In general, in non-relativistic theory ω_p can be an arbitrary function of p. It is worthwhile stressing that the dispersion relations (2.80) entering in the eigenvalue problem (2.79) are *exact*, i.e. they include all quantum corrections. In this subsection, to avoid discussing subtleties related to ultra-violet divergences, we assume that we are dealing with a lattice model.

To describe the *in*- and *out*-states, we introduce creation and annihilation *in*- and *out*-operators acting in the same Hilbert space and satisfying the canonical commutation relations (2.66). The Hilbert space has a state $|\Omega\rangle$, called vacuum, which is annihilated by all annihilation operators $a_{\rm in}(p,t)|\Omega\rangle = a_{\rm out}(p,t)|\Omega\rangle = 0$. The *in*- and *out*-states corresponding to free fields are obtained by applying creation *in*-operators $a_k^{\rm in\dagger}(p) \equiv a_k^{\rm in\dagger}(p,0)$ and *out*-operators $a_k^{\rm out\dagger}(p) \equiv a_k^{\rm out\dagger}(p,0)$ to the vacuum state, respectively,

$$\begin{aligned} |p_1,p_2,\ldots,p_n\rangle^{(in)}_{i_1,\ldots,i_n} &= a_{i_1}^{\rm in\dagger}(p_1)\cdots a_{i_n}^{\rm in\dagger}(p_n)|\Omega\rangle,\\ |p_1,p_2,\ldots,p_n\rangle^{(out)}_{i_1,\ldots,i_n} &= a_{i_1}^{\rm out\dagger}(p_1)\cdots a_{i_n}^{\rm out\dagger}(p_n)|\Omega\rangle. \end{aligned} \tag{2.81}$$

In the Heisenberg picture the time evolution of *in*- and *out*-operators is governed by the free Hamiltonians $\mathbb{H}^{\rm in}$ and $\mathbb{H}^{\rm out}$

$$\begin{aligned} \mathbb{H}^{\rm in} &= \int {\rm d}p \sum_i \omega_p^{(i)} a_i^{\rm in\dagger}(p) a^i_{\rm in}(p),\\ \mathbb{H}^{\rm out} &= \int {\rm d}p \sum_i \omega_p^{(i)} a_i^{\rm out\dagger}(p) a^i_{\rm out}(p). \end{aligned} \tag{2.82}$$

By construction, *in*/*out*-states (2.81) are the eigenstates of $\mathbb{H}_0^{\rm in/out}$ with the same eigenvalue (2.80).

The *in*- and *out*-operators satisfy the canonical commutation relations, and therefore, by virtue of the Stone–von Neumann theorem, they are related by a unitary operator $\mathbb{S}$

$$a^\dagger_{\rm in}(p,t) = \mathbb{S}\cdot a^\dagger_{\rm out}(p,t)\cdot\mathbb{S}^\dagger, \qquad a_{\rm in}(p,t) = \mathbb{S}\cdot a_{\rm out}(p,t)\cdot\mathbb{S}^\dagger, \qquad \mathbb{S}|\Omega\rangle = |\Omega\rangle, \tag{2.83}$$

which is the *S*-matrix operator. The *S*-matrix is time independent because the *in*- and *out*-operators have the same free field time dependence which factors out from equation (2.83). Therefore, *in* and *out* states are related as follows:

$$|p_1,\ldots,p_n\rangle^{(in)}_{i_1,\ldots,i_n} = \mathbb{S}\cdot|p_1,\ldots,p_n\rangle^{(out)}_{i_1,\ldots,i_n}, \tag{2.84}$$

and we can expand initial states on a basis of final states and vise versa. In particular, for the two-particle *in* and *out* states we get either

$$|p_1,p_2\rangle^{(in)}_{i,j} = \mathbb{S}\cdot|p_1,p_2\rangle^{(out)}_{i,j} = \mathbb{S}_{ij}^{kl}(p_1,p_2)|p_1,p_2\rangle^{(out)}_{k,l}, \tag{2.85}$$

or equivalently, by multiplying (2.85) by $\mathbb{S}$ and using (2.84)

$$\mathbb{S}\cdot|p_1,p_2\rangle^{(in)}_{i,j} = \mathbb{S}_{ij}^{kl}(p_1,p_2)|p_1,p_2\rangle^{(in)}_{k,l}. \tag{2.86}$$

Here we take into account that in one-dimensional space the set of momenta of the two scattering particles does not change in the scattering process, and we also order the particle momenta in decreasing order $p_1 > p_2 > \cdots > p_n$ to take into account the particle's statistics. It is clear that the S-matrix commutes with the full Hamiltonian

$$\mathbb{S} \cdot \mathbb{H} = \mathbb{H} \cdot \mathbb{S},$$

and that in the absence of interaction $\mathbb{S} = \mathbb{1}$, and $\mathbb{S}_{ij}^{kl}(p_1, p_2) = \delta_i^k \delta_j^l$. According to equations (2.85) and (2.86), it does not matter whether one computes the matrix elements $\mathbb{S}_{ij}^{kl}(p_1, p_2)$ by using the basis of in or out states.

If there is no external field, and the particles interact only among themselves, then a one-particle in-state coincides with its out-state, and therefore the S-matrix must also satisfy the following condition:

$$\mathbb{S}|p\rangle_k^{(\text{in})} = |p\rangle_k^{(\text{in})} \quad \Longleftrightarrow \quad \mathbb{S}|p\rangle_k^{(\text{out})} = |p\rangle_k^{(\text{out})}. \tag{2.87}$$

This condition can be used to determine dispersion relations.

To compute the S-matrix in perturbation theory one splits the full Hamiltonian into free and interaction parts

$$\mathbb{H} = \mathbb{H}_0 + \mathbb{V},$$

and introduces creation and annihilation operators $a, a^\dagger$ satisfying the canonical commutation relations (2.66). In terms of these operators the free Hamiltonian $\mathbb{H}_0$ takes the form

$$\mathbb{H}_0 = \int \mathrm{d}p \sum_k \omega_p^{(k)} a_k^\dagger(p, t) a^k(p, t). \tag{2.88}$$

The operators $a, a^\dagger$ (and $\mathbb{H}_0$) are interacting Heisenberg fields obeying the following equations of motion:

$$\dot{a}^k(p, t) = \mathrm{i}[\mathbb{H}, a^k(p, t)] = -\mathrm{i}\omega_p^{(k)} a^k(p, t) + \mathrm{i}[\mathbb{V}, a^k(p, t)], \tag{2.89}$$

where $\mathbb{V} = \mathbb{V}(a^\dagger, a)$ is a function of $a_k^\dagger$ and a^k. Note that if the dispersion relation receives quantum corrections then the interaction Hamiltonian $\mathbb{V}$ contains terms quadratic in $a, a^\dagger$.

Since the creation and annihilation operators $a, a^\dagger$ satisfy the canonical commutation relations, they are related to the in- and out-operators by unitary transformations

$$a^\dagger(p, t) = \mathbb{U}_{\text{in}}^\dagger(t) \cdot a_{\text{in}}^\dagger(p, t) \cdot \mathbb{U}_{\text{in}}(t), \qquad a(p, t) = \mathbb{U}_{\text{in}}^\dagger(t) \cdot a_{\text{in}}(p, t) \cdot \mathbb{U}_{\text{in}}(t), \tag{2.90}$$

$$a^\dagger(p, t) = \mathbb{U}_{\text{out}}(t) \cdot a_{\text{out}}^\dagger(p, t) \cdot \mathbb{U}_{\text{out}}^\dagger(t), \qquad a(p, t) = \mathbb{U}_{\text{out}}(t) \cdot a_{\text{out}}(p, t) \cdot \mathbb{U}_{\text{out}}^\dagger(t). \tag{2.91}$$

The unitary operators $\mathbb{U}_{\text{in}}, \mathbb{U}_{\text{out}}$ are determined up to constant unitary transformations, which we fix by imposing the following boundary conditions[19]:

$$\mathbb{U}_{\text{in}}(-\infty) = \mathbb{1}, \qquad \mathbb{U}_{\text{out}}(\infty) = \mathbb{1}, \tag{2.92}$$

and up to multiplication by a phase $\mathbb{U}(t) \to \mathrm{e}^{\mathrm{i}\varphi(t)}\mathbb{U}(t)$, where $\varphi(t)$ is an arbitrary real function independent of the creation and annihilation operators. In fact, the conditions (2.92) imply that the interacting Heisenberg field $a(p, t)$ tends to the free operators $a_{\text{in}}(p, t)$ and $a_{\text{out}}(p, t)$ in the asymptotic past $t \to -\infty$ and the asymptotic future $t \to +\infty$, respectively.

Comparing formulae (2.90) and (2.91) with equations (2.83) defining the S-matrix, we find the following expression for $\mathbb{S}$ in terms of $\mathbb{U}_{\text{in}}, \mathbb{U}_{\text{out}}$:

$$\mathbb{S} = \mathbb{U}_{\text{in}}(t) \cdot \mathbb{U}_{\text{out}}(t). \tag{2.93}$$

[19] It would be sufficient for our purposes to impose a weaker condition $\mathbb{U}_{\text{in}}(-\infty) \cdot \mathbb{U}_{\text{out}}(\infty) = \mathbb{1}$.

The S-matrix is time independent and, therefore, in the above formula we can put t to any desired value. Choosing $t = \infty$ or $t = -\infty$ and taking into account the boundary conditions (2.92), we get the following two convenient representation for the S-matrix:

$$\mathbb{S} = \mathbb{U}_{\rm in}(\infty) = \mathbb{U}_{\rm out}(-\infty). \tag{2.94}$$

To find $\mathbb{U}_{\rm in}$, we differentiate (2.90) with respect to t, and use the equations of motion for the operators involved. After simple algebra, we get the following equalities:

$$\begin{aligned} &\big[\dot{\mathbb{U}}_{\rm in}\mathbb{U}_{\rm in}^\dagger + {\rm i}\mathbb{V}\big(a_{\rm in}^\dagger, a_{\rm in}\big), a_{\rm in}^\dagger(p,t)\big] = 0, \\ &\big[\dot{\mathbb{U}}_{\rm in}\mathbb{U}_{\rm in}^\dagger + {\rm i}\mathbb{V}\big(a_{\rm in}^\dagger, a_{\rm in}\big), a_{\rm in}(p,t)\big] = 0, \end{aligned} \tag{2.95}$$

where the interaction Hamiltonian is now a function of the in-operators

$$\mathbb{V}\big(a_{\rm in}^\dagger, a_{\rm in}\big) = \mathbb{H}\big(a_{\rm in}^\dagger, a_{\rm in}\big) - \mathbb{H}_0^{\rm in} = \mathbb{U}_{\rm in}\mathbb{H}\big(a_k^\dagger, a^k\big)\mathbb{U}_{\rm in}^\dagger - \mathbb{H}_0^{\rm in}.$$

Equations (2.95) imply that $\dot{\mathbb{U}}_{\rm in}\mathbb{U}_{\rm in}^\dagger + {\rm i}\mathbb{V}\big(a_{\rm in}^\dagger, a_{\rm in}\big) = c(t)\mathbb{1}$, where $c(t)$ does not depend on $a_{\rm in}^\dagger, a_{\rm in}$. By properly choosing the phase $\varphi(t)$, we can always ensure vanishing of $c(t)$, so that $\mathbb{U}_{\rm in}$ will be then determined unambiguously by the following equation:

$$\dot{\mathbb{U}}_{\rm in}\mathbb{U}_{\rm in}^\dagger + {\rm i}\mathbb{V}\big(a_{\rm in}^\dagger, a_{\rm in}\big) = 0.$$

The equation can be solved in terms of the time-ordered exponential function $\mathcal{T}\exp$

$$\mathbb{U}_{\rm in}(t) = \mathcal{T}\exp\left(-{\rm i}\int_{-\infty}^{t} {\rm d}\tau\,\mathbb{V}\big(a_{\rm in}^\dagger(\tau), a_{\rm in}(\tau)\big)\right), \tag{2.96}$$

where we have taken into account the boundary condition (2.92) for $\mathbb{U}_{\rm in}$.

The operator $\mathbb{U}_{\rm out}$ can be found in the same way. It satisfies the equation

$$\mathbb{U}_{\rm out}^\dagger\dot{\mathbb{U}}_{\rm out} - {\rm i}\mathbb{V}\big(a_{\rm out}^\dagger, a_{\rm out}\big) = 0,$$

whose solution is given by the following formula:

$$\mathbb{U}_{\rm out}(t) = \mathcal{T}\exp\left(-{\rm i}\int_{t}^{\infty} {\rm d}\tau\,\mathbb{V}\big(a_{\rm out}^\dagger(\tau), a_{\rm out}(\tau)\big)\right). \tag{2.97}$$

Thus, we have derived the following two explicit expressions for the S-matrix:

$$\begin{aligned} \mathbb{S} &= \mathcal{T}\exp\left(-{\rm i}\int_{-\infty}^{\infty} {\rm d}\tau\,\mathbb{V}\big(a_{\rm in}^\dagger(\tau), a_{\rm in}(\tau)\big)\right) \\ &= \mathcal{T}\exp\left(-{\rm i}\int_{-\infty}^{\infty} {\rm d}\tau\,\mathbb{V}\big(a_{\rm out}^\dagger(\tau), a_{\rm out}(\tau)\big)\right). \end{aligned} \tag{2.98}$$

Expanding the formula in powers of $\mathbb{V}$, one develops the standard perturbation theory. We will need only the leading term in the expansion

$$\mathbb{S} = \mathbb{1} + {\rm i}\frac{1}{g}\mathbb{T}, \qquad \mathbb{T} = -g\int_{-\infty}^{\infty} {\rm d}\tau\,\mathbb{V}(\tau) + \cdots, \tag{2.99}$$

where $1/g$ is an expansion parameter of the perturbation theory.

This formula allows one to compute the world-sheet two-particle S-matrix for the light-cone string sigma model to the leading order in the $1/g$ expansion. To this end, one has to use the quadratic Hamiltonian (2.67) and the quartic Hamiltonian (2.56) as the free and interaction ones, respectively.

To complete our discussion of the general scattering theory, we note that in and out states can be also constructed in terms of the oscillators $a^\dagger(p) = a^\dagger(p, 0)$ and $a(p) = a(p, 0)$. Indeed, these oscillators are related to in and out operators through equations (2.90), (2.91)

$$a^\dagger(p) = \mathbb{U}_{\rm in}^\dagger(0)\cdot a_{\rm in}^\dagger(p)\cdot\mathbb{U}_{\rm in}(0), \qquad a(p) = \mathbb{U}_{\rm in}^\dagger(0)\cdot a_{\rm in}(p)\cdot\mathbb{U}_{\rm in}(0), \tag{2.100}$$

$$a^\dagger(p) = \mathbb{U}_{\rm out}(0)\cdot a^\dagger_{\rm out}(p)\cdot \mathbb{U}^\dagger_{\rm out}(0), \qquad a(p) = \mathbb{U}_{\rm out}(0)\cdot a_{\rm out}(p)\cdot \mathbb{U}^\dagger_{\rm out}(0). \tag{2.101}$$

As a result, we can write *in* and *out* states as follows:

$$|p_1, p_2, \ldots, p_n\rangle^{(in)}_{i_1,\ldots,i_n} = \mathbb{U}_{\rm in}(0)\cdot a^\dagger_{i_1}(p_1)\cdots a^\dagger_{i_n}(p_n)|0\rangle = \mathbb{U}_{\rm in}(0)|\Phi_\alpha\rangle,$$
$$|p_1, p_2, \ldots, p_n\rangle^{(out)}_{i_1,\ldots,i_n} = \mathbb{U}^\dagger_{\rm out}(0)\cdot a^\dagger_{i_1}(p_1)\cdots a^\dagger_{i_n}(p_n)|0\rangle = \mathbb{U}^\dagger_{\rm out}(0)|\Phi_\alpha\rangle,$$

where $|0\rangle = \mathbb{U}^\dagger_{\rm in}(0)|\Omega\rangle = \mathbb{U}_{\rm out}(0)|\Omega\rangle$ is the state annihilated by all operators $a(p)$: $a_k(p)|0\rangle = 0$, and α is a multi-index including all momenta and flavors of the scattering particles.

It is not difficult to find explicit formulae for the operators $\mathbb{U}_{\rm in}(0)$ and $\mathbb{U}_{\rm out}(0)$. To this end we introduce free time-dependent operators which have the same time dependence as the *in* and *out* operators

$$a^\dagger_{{\rm fr},k}(p,t) = {\rm e}^{{\rm i}\omega^{(k)}_p t} a^\dagger_k(p), \qquad a^k_{\rm fr}(p,t) = {\rm e}^{-{\rm i}\omega^{(k)}_p t} a^k(p).$$

The new oscillators are obviously related to *in* and *out* operators through the same equations (2.100), (2.101)

$$a^\dagger_{\rm fr}(p,t) = \mathbb{U}^\dagger_{\rm in}(0)\cdot a^\dagger_{\rm in}(p,t)\cdot \mathbb{U}_{\rm in}(0), \qquad a_{\rm fr}(p,t) = \mathbb{U}^\dagger_{\rm in}(0)\cdot a_{\rm in}(p,t)\cdot \mathbb{U}_{\rm in}(0),$$
$$a^\dagger_{\rm fr}(p,t) = \mathbb{U}_{\rm out}(0)\cdot a^\dagger_{\rm out}(p,t)\cdot \mathbb{U}^\dagger_{\rm out}(0), \qquad a_{\rm fr}(p,t) = \mathbb{U}_{\rm out}(0)\cdot a_{\rm out}(p,t)\cdot \mathbb{U}^\dagger_{\rm out}(0).$$

Thus, taking into account equations (2.96) and (2.97), we get the following formulae:

$$\mathbb{U}_{\rm in}(t) = \mathbb{U}_{\rm in}(0)\cdot \mathcal{T}\exp\left(-{\rm i}\int_{-\infty}^t {\rm d}\tau\, \mathbb{V}\big(a^\dagger_{\rm fr}(\tau), a_{\rm fr}(\tau)\big)\right)\cdot \mathbb{U}^\dagger_{\rm in}(0), \tag{2.102}$$

$$\mathbb{U}_{\rm out}(t) = \mathbb{U}^\dagger_{\rm out}(0)\cdot \mathcal{T}\exp\left(-{\rm i}\int_t^\infty {\rm d}\tau\, \mathbb{V}\big(a^\dagger_{\rm fr}(\tau), a_{\rm fr}(\tau)\big)\right)\cdot \mathbb{U}_{\rm out}(0). \tag{2.103}$$

From these expressions we can read off $\mathbb{U}_{\rm in}(0)$ and $\mathbb{U}_{\rm out}(0)$ in terms of the free oscillators $a^\dagger_{\rm fr}(\tau), a_{\rm fr}(\tau)$

$$\mathbb{U}_{\rm in}(0) = \mathcal{T}\exp\left(-\,{\rm i}\int_{-\infty}^0 {\rm d}\tau\, \mathbb{V}\big(a^\dagger_{\rm fr}(\tau), a_{\rm fr}(\tau)\big)\right), \tag{2.104}$$

$$\mathbb{U}_{\rm out}(0) = \mathcal{T}\exp\left(-{\rm i}\int_0^\infty {\rm d}\tau\, \mathbb{V}\big(a^\dagger_{\rm fr}(\tau), a_{\rm fr}(\tau)\big)\right). \tag{2.105}$$

Then we can easily find the overlap between *in* and *out* states, that is the *S*-matrix elements

$${}_\beta\langle out|in\rangle_\alpha = \langle\Phi_\beta|\mathbb{U}_{\rm out}(0)\mathbb{U}_{\rm in}(0)|\Phi_\alpha\rangle = \langle\Phi_\beta|\check{\mathbb{S}}|\Phi_\alpha\rangle,$$

where $\check{\mathbb{S}}$ is the following operator:

$$\check{\mathbb{S}} = \mathbb{U}_{\rm out}(0)\mathbb{U}_{\rm in}(0) = \mathcal{T}\exp\left(-{\rm i}\int_{-\infty}^\infty {\rm d}\tau\, \mathbb{V}\big(a^\dagger_{\rm fr}(\tau), a_{\rm fr}(\tau)\big)\right),$$

Note that the operator $\check{\mathbb{S}}$ differs from the *S*-matrix operator $\mathbb{S}$ in equation (2.93) by the opposite order of $\mathbb{U}_{\rm in}(0)$ and $\mathbb{U}_{\rm out}(0)$.

It is not difficult to show that the operators $\mathbb{U}_{\rm in}(0)$ and $\mathbb{U}_{\rm out}(0)$ have the following commutation relations with $\mathbb{H}$ and $\mathbb{H}_0(0)$:

$$\mathbb{H}\cdot \mathbb{U}_{\rm in}(0) = \mathbb{U}_{\rm in}(0)\cdot \mathbb{H}_0(0), \qquad \mathbb{H}_0(0)\cdot \mathbb{U}_{\rm out}(0) = \mathbb{U}_{\rm out}(0)\cdot \mathbb{H},$$

and, therefore, the operator $\mathbb{H}_0(0)$ commutes with $\check{\mathbb{S}}$

$$\mathbb{H}_0(0)\cdot \check{\mathbb{S}} = \check{\mathbb{S}}\cdot \mathbb{H}_0(0).$$

2.3.2. A sample computation of perturbative S-matrix. To illustrate how the formulae above can be used, let us compute the perturbative S-matrix for the $Y^{a\dot a}$ bosons from the 5-sphere. The relevant part of the T-matrix operator is given by

$$\mathbb{T}_Y = -g\int_{-\infty}^{\infty} \mathrm{d}\tau\, \mathbb{V}(\tau) = 2\int_{-\infty}^{\infty} \mathrm{d}\tau\, \mathrm{d}\sigma\, Y^{a\dot a} Y_{a\dot a} Y'^{b\dot b} Y'_{b\dot b}, \tag{2.106}$$

where we used equation (2.58) for the quartic Hamiltonian, and lowered the indices by means of the ϵ-tensor $Y_{a\dot a} = \epsilon_{ab}\epsilon_{\dot a\dot b} Y^{b\dot b}$. We use the mode decomposition (2.64) with the creation and annihilation operators having the free-field time dependence

$$a^{a\dot a}(p,t) = \mathrm{e}^{-\mathrm{i}\omega_p t} a^{a\dot a}(p), \qquad a^\dagger_{a\dot a}(p,t) = \mathrm{e}^{\mathrm{i}\omega_p t} a^\dagger_{a\dot a}(p).$$

The creation and annihilation operators are either in or out-operators depending on the basis we use for the S-matrix computation.

Substituting the mode decomposition into (2.106), and integrating over τ and σ, one gets a sum of terms of the form

$$\begin{aligned}
&\delta(\omega_1+\omega_2+\omega_3+\omega_4)\delta(k_1+k_2+k_3+k_4)a^\dagger(k_1)a^\dagger(k_2)a^\dagger(k_3)a^\dagger(k_4)\\
&\quad+\delta(\omega_1+\omega_2+\omega_3-\omega_4)\delta(k_1+k_2+k_3-k_4)a^\dagger(k_1)a^\dagger(k_2)a^\dagger(k_3)a(k_4)\\
&\quad+\delta(\omega_1+\omega_2-\omega_3-\omega_4)\delta(k_1+k_2-k_3-k_4)a^\dagger(k_1)a^\dagger(k_2)a(k_3)a(k_4)+h.c.
\end{aligned}$$

One can easily check that due to the energy/momentum conservation delta-functions only the terms with equal number of creation and annihilation operators do not vanish. Then, a simple computation gives

$$\begin{aligned}
\mathbb{T}_Y = \int &\frac{\mathrm{d}k_1\,\mathrm{d}k_2\,\mathrm{d}k_3\,\mathrm{d}k_4}{4\sqrt{\omega_1\omega_2\omega_3\omega_4}}\delta(\omega_1+\omega_2-\omega_3-\omega_4)\delta(k_1+k_2-k_3-k_4)\\
&\times\big[(2k_2k_4-k_1k_2-k_3k_4)a^\dagger_{b\dot b}(k_4)a^\dagger_{a\dot a}(k_3)a^{b\dot b}(k_2)a^{a\dot a}(k_1)\\
&+(k_1k_2+k_3k_4)a^\dagger_{a\dot b}(k_4)a^\dagger_{b\dot a}(k_3)a^{b\dot b}(k_2)a^{a\dot a}(k_1)\big].
\end{aligned}$$

The δ-functions can be used to integrate over k_3 and k_4 because they imply that either $k_3 = k_1, k_4 = k_2$ or $k_3 = k_2, k_4 = k_1$, and taking into account that the Jacobian of $\delta(\omega_1+\omega_2-\omega_3-\omega_4)$ equals $\omega_1\omega_2/|k_1\omega_2-k_2\omega_1|$, one gets the T-matrix

$$\begin{aligned}
\mathbb{T}_Y = \int \frac{\mathrm{d}k_1\,\mathrm{d}k_2}{2|k_1\omega_2-k_2\omega_1|}\bigg[&\frac{1}{2}(k_1-k_2)^2 a^\dagger_{b\dot b}(k_2)a^\dagger_{a\dot a}(k_1)a^{b\dot b}(k_2)a^{a\dot a}(k_1)\\
&+2k_1k_2 a^\dagger_{a\dot b}(k_2)a^\dagger_{b\dot a}(k_1)a^{b\dot b}(k_2)a^{a\dot a}(k_1)\bigg].
\end{aligned}$$

Finally, acting by the T-matrix operator on a two-particle state, one derives

$$\begin{aligned}
\mathbb{T}_Y\cdot\big|a^\dagger_{a\dot a}(p_1)a^\dagger_{b\dot b}(p_2)\big\rangle &= \frac{(p_1-p_2)^2}{2(p_1\omega_2-p_2\omega_1)}\big|a^\dagger_{a\dot a}(p_1)a^\dagger_{b\dot b}(p_2)\big\rangle\\
&\quad+\frac{p_1p_2}{p_1\omega_2-p_2\omega_1}\big(\big|a^\dagger_{b\dot a}(p_1)a^\dagger_{a\dot b}(p_2)\big\rangle+\big|a^\dagger_{a\dot b}(p_1)a^\dagger_{b\dot a}(p_2)\big\rangle\big),
\end{aligned}\tag{2.107}$$

where we have assumed that $p_1 > p_2$.

The action of the T-matrix operator on an arbitrary two-particle state is given in appendix 2.5.3.

2.3.3. S-matrix factorization. Formula (2.107) for the T-matrix shows that it has the following factorized form:

$$\mathbb{T}_Y = \mathcal{T}_Y\otimes\mathbb{1}+\mathbb{1}\otimes\dot{\mathcal{T}}_Y,$$

where the operators $\mathcal{T}_Y$ and $\dot{\mathcal{T}}_Y$ act only on the undotted and dotted indices, respectively. Moreover, analyzing the formulae from appendix 2.5.3, one can show that the same factorization also holds for the full T-matrix: $\mathbb{T} = \mathcal{T} \otimes \mathbb{1} + \mathbb{1} \otimes \dot{\mathcal{T}}$. This factorization in fact follows from the corresponding factorization of the S-matrix operator:

$$\mathbb{S} = \mathcal{S} \otimes \dot{\mathcal{S}},$$

which is a consequence of the integrability of the model, as will be discussed in the following section in detail.

The simplest way to describe the factorization is to think about the two-index creation operators $a^\dagger_{M\dot{M}}$ as a product of two one-index operators $a^\dagger_M$ and $a^\dagger_{\dot{M}}$, that is $a^\dagger_{M\dot{M}}(p) \sim a^\dagger_M(p)a^\dagger_{\dot{M}}(p)$. Since the lower-case Latin indices are even, and the Greek indices are odd, the operators $a^\dagger_a, a^\dagger_{\dot{a}}$ are bosonic, and $a^\dagger_\alpha, a^\dagger_{\dot{\alpha}}$ are fermionic, and they commute or anti-commute depending on their statistics.

We see, therefore, that one-particle states can be identified with the following tensor product:

$$\left|a^\dagger_{M\dot{M}}(p)\right\rangle \sim \left|a^\dagger_M(p)\right\rangle \otimes \left|a^\dagger_{\dot{M}}(p)\right\rangle,$$

and two-particle states with

$$\left|a^\dagger_{M\dot{M}}(p_1)a^\dagger_{N\dot{N}}(p_2)\right\rangle \sim (-1)^{\epsilon_{\dot{M}}\epsilon_N}\left|a^\dagger_M(p_1)a^\dagger_N(p_2)\right\rangle \otimes \left|a^\dagger_{\dot{M}}(p_1)a^\dagger_{\dot{N}}(p_2)\right\rangle, \quad (2.108)$$

where the extra sign may appear because one permutes the operators $a^\dagger_{\dot{M}}$ and $a^\dagger_N$.

Then, $\mathcal{S}$ and $\dot{\mathcal{S}}$ act in the space of the $\left|a^\dagger_M(p_1)a^\dagger_N(p_2)\right\rangle$ and $\left|a^\dagger_{\dot{M}}(p_1)a^\dagger_{\dot{N}}(p_2)\right\rangle$ states, respectively, and their S-matrix elements are defined in the usual way

$$\mathcal{S}\cdot\left|a^\dagger_M(p_1)a^\dagger_N(p_2)\right\rangle = \mathcal{S}^{PQ}_{MN}(p_1,p_2)\left|a^\dagger_P(p_1)a^\dagger_Q(p_2)\right\rangle, \quad (2.109)$$

and a similar formula for $\dot{\mathcal{S}}$. In particular, we find from (2.107) the action of $\mathcal{T}_Y$ on the states

$$\mathcal{T}_Y\cdot\left|a^\dagger_a(p_1)a^\dagger_b(p_2)\right\rangle = \frac{(p_1-p_2)^2}{4(p_1\omega_2-p_2\omega_1)}\left|a^\dagger_a(p_1)a^\dagger_b(p_2)\right\rangle + \frac{p_1p_2}{p_1\omega_2-p_2\omega_1}\left|a^\dagger_b(p_1)a^\dagger_a(p_2)\right\rangle.$$

By using (2.108) and (2.109), one can easily derive the following relation between the elements of the scattering matrix $\mathbb{S}$, and those of the auxiliary S-matrices $\mathcal{S}$ and $\dot{\mathcal{S}}$:

$$\mathbb{S}^{P\dot{P},Q\dot{Q}}_{M\dot{M},N\dot{N}}(p_1,p_2) = (-1)^{\epsilon_{\dot{M}}\epsilon_N+\epsilon_{\dot{P}}\epsilon_Q}\mathcal{S}^{PQ}_{MN}(p_1,p_2)\dot{\mathcal{S}}^{\dot{P}\dot{Q}}_{\dot{M}\dot{N}}(p_1,p_2). \quad (2.110)$$

Taking into account that

$$\mathbb{S} = \mathbb{1} + \mathrm{i}\frac{1}{g}\mathbb{T}, \qquad \mathcal{S} = \mathbb{1} + \mathrm{i}\frac{1}{g}\mathcal{T}, \qquad \dot{\mathcal{S}} = \mathbb{1} + \mathrm{i}\frac{1}{g}\dot{\mathcal{T}},$$

one finds the following relation:

$$\mathbb{T}^{P\dot{P},Q\dot{Q}}_{M\dot{M},N\dot{N}} = (-1)^{\epsilon_{\dot{M}}(\epsilon_N+\epsilon_Q)}\mathcal{T}^{PQ}_{MN}\delta^{\dot{P}}_{\dot{M}}\delta^{\dot{Q}}_{\dot{N}} + (-1)^{(\epsilon_{\dot{M}}+\epsilon_{\dot{P}})\epsilon_Q}\delta^P_M\delta^Q_N\dot{\mathcal{T}}^{\dot{P}\dot{Q}}_{\dot{M}\dot{N}} \quad (2.111)$$

for the T-matrix elements. The matrix elements for $\mathcal{T}$ and $\dot{\mathcal{T}}$ can be chosen to be equal to each other, and can be extracted from the formulae in appendix 2.5.3. The result can be written in the following form:

$$\begin{aligned}
\mathcal{T}^{cd}_{ab} &= A\delta^c_a\delta^d_b + B\delta^d_a\delta^c_b, & \mathcal{T}^{\gamma\delta}_{ab} &= C\epsilon_{ab}\epsilon^{\gamma\delta},\\
\mathcal{T}^{\gamma\delta}_{\alpha\beta} &= D\delta^\gamma_\alpha\delta^\delta_\beta + E\delta^\delta_\alpha\delta^\gamma_\beta, & \mathcal{T}^{cd}_{\alpha\beta} &= F\epsilon_{\alpha\beta}\epsilon^{cd},\\
\mathcal{T}^{c\delta}_{a\beta} &= G\delta^c_a\delta^\delta_\beta, & \mathcal{T}^{\gamma d}_{\alpha b} &= L\delta^\gamma_\alpha\delta^d_b,\\
\mathcal{T}^{\gamma d}_{a\beta} &= H\delta^d_a\delta^\gamma_\beta, & \mathcal{T}^{\gamma d}_{\alpha b} &= K\delta^\gamma_\alpha\delta^d_b
\end{aligned}$$

where the coefficients are given by

$$\begin{aligned}
A(p_1,p_2) &= \frac{(p_1-p_2)^2}{4(p_1\omega_2-p_2\omega_1)}+\frac{1}{4}(1-2a)(p_1\omega_2-p_2\omega_1),\\
B(p_1,p_2) &= -E(p_1,p_2)=\frac{p_1p_2}{p_1\omega_2-p_2\omega_1},\\
C(p_1,p_2) &= F(p_1,p_2)=\frac{1}{2}\frac{\sqrt{(\omega_1+1)(\omega_2+1)}(p_1\omega_2-p_2\omega_1+p_2-p_1}{p_1\omega_2-p_2\omega_1},\\
D(p_1,p_2) &= -\frac{(p_1-p_2)^2}{4(p_1\omega_2-p_2\omega_1)}+\frac{1}{4}(1-2a)(p_1\omega_2-p_2\omega_1),\\
G(p_1,p_2) &= -L(p_2,p_1)=-\frac{p_1^2-p_2^2}{4(p_1\omega_2-p_2\omega_1)}+\frac{1}{4}(1-2a)(p_1\omega_2-p_2\omega_1),\\
H(p_1,p_2) &= K(p_1,p_2)=\frac{1}{2}\frac{p_1p_2}{p_1\omega_2-p_2\omega_1}\frac{(\omega_1+1)(\omega_2+1)-p_1p_2}{\sqrt{(\omega_1+1)(\omega_2+1)}},
\end{aligned} \tag{2.112}$$

where we have also added the additional contribution which vanishes in the $a=1/2$ gauge. The T-matrix $\mathcal{T}$ is covariant under the SU(2)× SU(2) transformations that reflect the manifest $\mathrm{SU}(2)^4$ symmetry of the light-cone string sigma model. The factorization of the T-matrix is a non-trivial test of the integrability of the model.

2.4. Symmetry algebra

In this section we show that the symmetry algebra of the light-cone string sigma model in the decompactification limit gets enlarged by two additional central charges which vanish on the physical subspace of the model.

2.4.1. General structure of symmetry generators.

The invariance of the Green–Schwarz action under the group PSU(2,2|4) leads to the existence of conserved currents and charges. As was shown in the previous section, see equation (1.54), the conserved currents can be written in terms of A_α as follows:

$$J^\alpha = g\mathfrak{g}(x,\chi)\Big(\gamma^{\alpha\beta}A_\beta^{(2)}-\frac{\kappa}{2}\epsilon^{\alpha\beta}\big(A_\beta^{(1)}-A_\beta^{(3)}\big)\Big)\mathfrak{g}(x,\chi)^{-1}. \tag{2.113}$$

The 8×8 supermatrix Q of conserved charges is then given by the integral over σ of J^τ, equation (1.56). For our purposes it is convenient to express the charges in terms of the momentum π. To this end, we note that, as follows from (2.18), π satisfies the following equation of motion:

$$\pi = g\gamma^{\tau\beta}A_\beta^{(2)} = g\gamma^{\tau\tau}\left(A_\tau^{(2)}+\frac{\gamma^{\tau\sigma}}{\gamma^{\tau\tau}}A_\sigma^{(2)}\right). \tag{2.114}$$

Therefore, we can express $A_\tau^{(2)}$ in terms of π, and substitute it into the expression for Q. After some simple algebra we get

$$\mathrm{Q}=\int_{-r}^{r}\mathrm{d}\sigma\,\mathfrak{g}(x,\chi)\left(\pi-g\frac{\kappa}{2}\big(A_\sigma^{(1)}-A_\sigma^{(3)}\big)\right)\mathfrak{g}(x,\chi)^{-1}.$$

The formula can be written in a more explicit form if we take into account that

$$A_\sigma^{(1)}-A_\sigma^{(3)}=\mathrm{i}\mathfrak{g}(x)\mathcal{K}F_\sigma^{\mathrm{st}}\mathcal{K}\mathfrak{g}(x)^{-1},$$

where F_σ is an odd component of the current $g^{-1}(\chi)\partial_\sigma g(\chi)$ defined in (2.33). Then, the $\mathfrak{psu}(2,2|4)$ charges are

$$Q = \int_{-r}^{r} d\sigma\, \Lambda \mathfrak{g}(\chi)\mathfrak{g}(x)\left(\pi - \mathrm{i}g\frac{\kappa}{2}\mathfrak{g}(x)\mathcal{K}F_\sigma^{\mathrm{st}}\mathcal{K}\mathfrak{g}(x)^{-1}\right)\mathfrak{g}(x)^{-1}\mathfrak{g}(\chi)^{-1}\Lambda^{-1}. \tag{2.115}$$

The expression is very simple and it has an important property of being explicitly independent of the world-sheet metric.

We also see that the matrix Q can be schematically written as follows:

$$Q = \int_{-r}^{r} d\sigma\, \Lambda U \Lambda^{-1}, \tag{2.116}$$

where U depends on physical fields (x, p, χ) but not on $x_\pm$ and, therefore, is a local function of σ. The only dependence of Q on $x_\pm$ occurs through the matrix Λ (1.121) which has the following form in the $a = 1/2$ light-cone gauge

$$\Lambda = \mathrm{e}^{\frac{\mathrm{i}}{2}x_+\Sigma_+ + \frac{\mathrm{i}}{4}x_-\Sigma_-}, \tag{2.117}$$

where $\Sigma_\pm$ are defined in (2.22), and $x_+ = \tau$ due to the light-cone gauge condition.

We recall that the field x_- is unphysical and can be solved in terms of physical excitations through the equation

$$x'_- = -\frac{1}{g}\left(p_M x'_M - \frac{\mathrm{i}}{2}\,\mathrm{str}(\Sigma_+\chi\chi')\right) + \cdots, \tag{2.118}$$

where $\cdots$ denote terms which are of higher order in the fields. This equation determines x_- up to a function of τ which is the zero mode of x_- canonically conjugated to P_+. The τ-dependence of the zero mode can be determined from the evolution equation for x_-. In what follows we need to know the symmetry algebra generators in the decompactification limit only. In this limit the Hamiltonian and the symmetry generators do not depend on P_+ and, for this reason, the zero mode becomes a central element.

Linear combinations of components of the matrix Q produce charges which generate rotations, dilatation, supersymmetry and so on. To single them out one should multiply Q by a corresponding 8×8 matrix $\mathcal{M}$, and take the supertrace

$$Q_{\mathcal{M}} = \mathrm{str}(Q\mathcal{M}). \tag{2.119}$$

It is clear that the diagonal and off-diagonal 4×4 blocks of $\mathcal{M}$ single out bosonic and fermionic charges of $\mathfrak{psu}(2,2|4)$, respectively. In particular, one can check that the light-cone Hamiltonian can be obtained from Q as follows:

$$H = -\frac{\mathrm{i}}{2}\,\mathrm{str}(Q\Sigma_+), \tag{2.120}$$

and the light-cone momentum P_+ is given by

$$P_+ = \frac{\mathrm{i}}{4}\,\mathrm{str}(Q\Sigma_-). \tag{2.121}$$

Depending on the choice of $\mathcal{M}$ the charges $Q_{\mathcal{M}} \equiv Q_{\mathcal{M}}(x_+, x_-)$ can be naturally classified according to their dependence on $x_\pm$. First, with respect to x_- they are divided into *kinematical* (independent of x_-) and *dynamical* (dependent on x_-). Kinematical generators do not receive quantum corrections, while the dynamical generators do. Second, the charges, both kinematical and dynamical, may or may not explicitly depend on $x_+ = \tau$.

In the Hamiltonian setting the conservation laws have the following form:

$$\frac{dQ_{\mathcal{M}}}{d\tau} = \frac{\partial Q_{\mathcal{M}}}{\partial\tau} + \{H, Q_{\mathcal{M}}\} = 0.$$

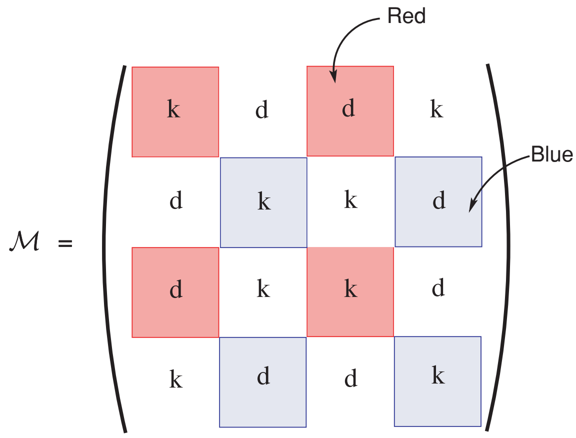

Figure 3. The distribution of the kinematical and dynamical charges in the $\mathcal{M}$ supermatrix. The red (dark) and blue (light) blocks correspond to the subalgebra $\mathcal{J}$ of $\mathfrak{psu}(2,2|4)$ which leaves the Hamiltonian invariant.

Therefore, the generators which do not have explicit dependence on $x_+ = \tau$ Poisson-commute with the Hamiltonian. As follows from the Jacobi identity, they must form an algebra which contains H as central element.

Analyzing the structure of Q one can establish how a generic matrix $\mathcal{M}$ is split into 2×2 blocks each of them giving rise either to kinematical or dynamical generators. This splitting of $\mathcal{M}$ is shown in figure 3, where the kinematical blocks are denoted by **k** and the dynamical ones by **d**, respectively. Furthermore, one can see that the blocks which are colored in red and blue give rise to charges which are independent of $x_+ = \tau$; by this reason these charges commute with the Hamiltonian and form the manifest symmetry algebra of the gauge-fixed string sigma model. Complementary, we note that the uncolored both kinematical (fermionic) and dynamical (bosonic) generators do depend on x_+.

These conclusions about the structure of $\mathcal{M}$ can be easily drawn by noting that Λ in equation (2.117) is built out of two commuting matrices Σ_+ and Σ_-. For instance, leaving in $\mathcal{M}$ the kinematical blocks only, i.e. $\mathcal{M} \equiv \mathcal{M}_{\rm kin}$, we find that $[\Sigma_-, \mathcal{M}_{\rm kin}] = 0$ and, therefore, due to the structure of $Q_{\mathcal{M}}$, see equation (2.119), the variable x_- cancels out in $Q_{\mathcal{M}}$. On the other hand, any matrix from the red–blue submatrix $\mathcal{J}$ of $\mathcal{M}$ in figure 3 commutes with the element Σ_+ in $\mathfrak{psu}(2,2|4)$

$$[\Sigma_+, \mathcal{M}] = 0, \mathcal{M} \in \mathcal{J},$$

leading to a charge $Q_{\mathcal{M}}$ independent of $x_+ = \tau$. Thus, for P_+ finite we obtain the following vector space decomposition of $\mathcal{J}$:

$$\mathcal{J} = \mathfrak{psu}(2|2) \oplus \mathfrak{psu}(2|2) \oplus \Sigma_+ \oplus \Sigma_-.$$

The rank of the latter subalgebra is six and it coincides with that of $\mathfrak{psu}(2,2|4)$. In the case of infinite P_+ the last generator decouples.

Conjugating with Λ of (2.117), one finds

$$\begin{aligned} \Lambda^{-1}\mathcal{M}^{\rm odd}_{\rm dyn}\Lambda &= {\rm e}^{-\frac{\rm i}{2}x_-\Sigma_-}\mathcal{M}^{\rm odd}_{\rm dyn}, & \Lambda^{-1}\mathcal{M}^{\rm even}_{\rm dyn}\Lambda &= \Lambda^2\mathcal{M}^{\rm even}_{\rm dyn}, \\ \Lambda^{-1}\mathcal{M}^{\rm odd}_{\rm kin}\Lambda &= {\rm e}^{{\rm i}x_+\Sigma_+}\mathcal{M}^{\rm odd}_{\rm kin}, & \Lambda^{-1}\mathcal{M}^{\rm even}_{\rm kin}\Lambda &= \mathcal{M}^{\rm even}_{\rm kin}, \end{aligned} \qquad (2.122)$$

which shows that the $x_+ = \tau$ independent matrices are indeed given by $\mathcal{M}_{\rm dyn}^{\rm odd}$ and $\mathcal{M}_{\rm kin}^{\rm even}$, i.e. by the red and blue entries in figure 3. We see from figure 3 and (2.122) that in the symmetry algebra all bosonic charges are kinematic, and all supercharges are dynamical.

The structure of Q discussed above is found for finite r and it also remains valid in the decompactification limit $r \to \infty$.

2.4.2. Centrally extended $\mathfrak{su}(2|2)$ *algebra.* It is clear that the $\mathfrak{psu}(2,2|4)$ charges (2.115) transform linearly under the bosonic subalgebra $\mathfrak{C}$ defined in (1.127) because Λ commutes with any element of this subalgebra. Therefore, to encode the transformation properties of the charges under $\mathfrak{C}$, it is convenient to use the two-index notation introduced in section 1.4. The time-dependent charges in the white blocks of figure 3 have the same indices as the bosonic and fermionic fields $Z^{\alpha\dot\alpha}$, $Y^{a\dot a}$, $\theta^{a\dot\alpha}$, $\eta^{a\dot\alpha}$. The time-independent charges which commute with the Hamiltonian and form the symmetry algebra can be represented in terms of 2×2 blocks as follows:

$$\mathrm{Q}_{\rm sym} = \begin{pmatrix} \mathbb{R} & 0 & -\mathbb{Q}^\dagger & 0 \\ 0 & \mathring{\mathbb{R}} & 0 & \mathring{\mathbb{Q}} \\ \mathbb{Q} & 0 & \mathbb{L} & 0 \\ 0 & \mathring{\mathbb{Q}}^\dagger & 0 & \mathring{\mathbb{L}} \end{pmatrix}. \tag{2.123}$$

Here $\mathbb{R}, \mathring{\mathbb{R}} \in \mathfrak{su}(2,2)$, and $\mathbb{L}, \mathring{\mathbb{L}} \in \mathfrak{su}(4)$ are the bosonic charges which generate the transformations under $\mathfrak{C}$, and $\mathbb{Q}, \mathbb{Q}^\dagger, \mathring{\mathbb{Q}}, \mathring{\mathbb{Q}}^\dagger$ are the eight complex supercharges. The bosonic charges satisfy the usual reality and tracelessness conditions

$$\begin{gathered} \mathbb{R}^\dagger = -\mathbb{R}, \qquad \mathring{\mathbb{R}}^\dagger = -\mathring{\mathbb{R}}, \qquad \mathbb{L}^\dagger = -\mathbb{L}, \qquad \mathring{\mathbb{L}}^\dagger = -\mathring{\mathbb{L}}, \\ \mathrm{tr}\mathbb{R} = \mathrm{tr}\mathring{\mathbb{R}} = \mathrm{tr}\mathbb{L} = \mathrm{tr}\mathring{\mathbb{L}} = 0. \end{gathered} \tag{2.124}$$

These charges should be complemented by the matrices representing the Hamiltonian and the light-cone momentum which are of the form

$$\mathrm{Q}_{\mathbb{H}} = -\frac{\mathrm{i}}{4}\mathbb{H}\begin{pmatrix} -\mathbb{1} & 0 & 0 & 0 \\ 0 & \mathbb{1} & 0 & 0 \\ 0 & 0 & \mathbb{1} & 0 \\ 0 & 0 & 0 & -\mathbb{1} \end{pmatrix}, \qquad \mathrm{Q}_{\mathbb{P}_+} = \frac{\mathrm{i}}{2}\mathbb{P}_+\begin{pmatrix} \mathbb{1} & 0 & 0 & 0 \\ 0 & -\mathbb{1} & 0 & 0 \\ 0 & 0 & \mathbb{1} & 0 \\ 0 & 0 & 0 & -\mathbb{1} \end{pmatrix}. \tag{2.125}$$

In our analysis the light-cone momentum will not play any role because we will only discuss the decompactification limit where $P_+ \to \infty$.

Under the action of the group element (1.128) the matrix (2.123) transforms as follows:

$$\mathrm{Q}_{\rm sym} \to G\,\mathrm{Q}_{\rm sym}G^{-1} = \begin{pmatrix} \mathfrak{g}_1\mathbb{R}\mathfrak{g}_1^{-1} & 0 & -\mathfrak{g}_1\mathbb{Q}^\dagger\mathfrak{g}_3^{-1} & 0 \\ 0 & \mathfrak{g}_2\mathring{\mathbb{R}}\mathfrak{g}_2^{-1} & 0 & \mathfrak{g}_2\mathring{\mathbb{Q}}\mathfrak{g}_4^{-1} \\ \mathfrak{g}_3\mathbb{Q}\mathfrak{g}_1^{-1} & 0 & \mathfrak{g}_3\mathbb{L}\mathfrak{g}_3^{-1} & 0 \\ 0 & \mathfrak{g}_4\mathring{\mathbb{Q}}^\dagger\mathfrak{g}_2^{-1} & 0 & \mathfrak{g}_4\mathring{\mathbb{L}}\mathfrak{g}_4^{-1} \end{pmatrix}. \tag{2.126}$$

Since the charges $\mathbb{R}, \mathbb{L}, \mathbb{Q}, \mathbb{Q}^\dagger$ transform under one $\mathfrak{su}(2) \in \mathfrak{su}(2,2)$ and one $\mathfrak{su}(2) \in \mathfrak{su}(4)$, and the charges $\mathring{\mathbb{R}}, \mathring{\mathbb{L}}, \mathring{\mathbb{Q}}, \mathring{\mathbb{Q}}^\dagger$ transform under another $\mathfrak{su}(2) \in \mathfrak{su}(2,2)$ and another $\mathfrak{su}(2) \in \mathfrak{su}(4)$, the charges from the first group must (anti-)commute with those from the second group.

Repeating the considerations in subsection 1.4.2, we find that the 2×2 blocks $\mathbb{R}$, $\mathbb{L}$, $\mathbb{Q}$, $\mathbb{Q}^\dagger$ are expressed via covariant two-index entries $\mathbb{L}^{ab}$, $\mathbb{R}^{\alpha\beta}$, $\mathbb{Q}^{\alpha b}$, $\mathbb{Q}^\dagger_{a\beta}$ as

$$\mathbb{L} = -\,\mathrm{i}\begin{pmatrix}\mathbb{L}^{12} & -\mathbb{L}^{11}\\ \mathbb{L}^{22} & -\mathbb{L}^{21}\end{pmatrix}, \qquad \mathbb{R} = \,\mathrm{i}\begin{pmatrix}\mathbb{R}^{34} & -\mathbb{R}^{33}\\ \mathbb{R}^{44} & -\mathbb{R}^{43}\end{pmatrix},$$
$$\mathbb{Q} = \mathrm{e}^{\mathrm{i}\pi/4}\begin{pmatrix}\mathbb{Q}^{41} & -\mathbb{Q}^{31}\\ \mathbb{Q}^{42} & -\mathbb{Q}^{32}\end{pmatrix}, \qquad \mathbb{Q}^\dagger = \mathrm{e}^{-\mathrm{i}\pi/4}\begin{pmatrix}\mathbb{Q}^\dagger_{14} & \mathbb{Q}^\dagger_{24}\\ -\mathbb{Q}^\dagger_{13} & -\mathbb{Q}^\dagger_{23}\end{pmatrix},$$

and $\mathring{\mathbb{R}}$, $\mathring{\mathbb{L}}$, $\mathring{\mathbb{Q}}$, $\mathring{\mathbb{Q}}^\dagger$ are expressed through $\mathbb{L}^{\dot a\dot b}$, $\mathbb{R}^{\dot\alpha\dot\beta}$, $\mathbb{Q}^{\dot\alpha\dot b}$, $\mathbb{Q}^\dagger_{\dot a\dot\beta}$ as

$$\mathring{\mathbb{L}} = -\mathrm{i}\begin{pmatrix}\mathbb{L}^{\dot1\dot2} & -\mathbb{L}^{\dot1\dot1}\\ \mathbb{L}^{\dot2\dot2} & -\mathbb{L}^{\dot2\dot1}\end{pmatrix}, \qquad \mathring{\mathbb{R}} = \,\mathrm{i}\begin{pmatrix}\mathbb{R}^{\dot3\dot4} & -\mathbb{R}^{\dot3\dot3}\\ \mathbb{R}^{\dot4\dot4} & -\mathbb{R}^{\dot4\dot3}\end{pmatrix},$$
$$\mathring{\mathbb{Q}} = -\mathrm{e}^{\mathrm{i}\pi/4}\begin{pmatrix}\mathbb{Q}^{\dot3\dot2} & -\mathbb{Q}^{\dot3\dot1}\\ \mathbb{Q}^{\dot4\dot2} & -\mathbb{Q}^{\dot4\dot1}\end{pmatrix}, \qquad \mathring{\mathbb{Q}}^\dagger = -\mathrm{e}^{-\mathrm{i}\pi/4}\begin{pmatrix}\mathbb{Q}^\dagger_{\dot2\dot3} & \mathbb{Q}^\dagger_{\dot2\dot4}\\ -\mathbb{Q}^\dagger_{\dot1\dot3} & -\mathbb{Q}^\dagger_{\dot1\dot4}\end{pmatrix}.$$

Here, by definition, $\mathbb{Q}^\dagger_{a\beta}$ and $\mathbb{Q}^\dagger_{\dot a\dot\beta}$ are understood as Hermitian conjugate of $\mathbb{Q}^{\beta a}$ and $\mathbb{Q}^{\dot\beta\dot a}$, respectively,

$$(\mathbb{Q}^{\beta a})^\dagger = \mathbb{Q}^\dagger_{a\beta}, \qquad (\mathbb{Q}^{\dot\beta\dot a})^\dagger = \mathbb{Q}^\dagger_{\dot a\dot\beta},$$

and the tracelessness condition for bosonic charges implies that they are symmetric: $\mathbb{L}^{ab} = \mathbb{L}^{ba}$ and so on. Note also that according to the transformation rule (2.126) for $\mathbb{Q}$, it would be more consistent to write the entries of $\mathbb{Q}$ as $\mathbb{Q}^{b\alpha}$ rather than $\mathbb{Q}^{\alpha b}$. However, the order of the indices does not matter because the transformations by the group elements $\mathfrak{g}_1$ and $\mathfrak{g}_3$ are independent, and with the choice we made many formulae for the dotted operators are obtained from the undotted ones by replacing correspondingly the indices. The phases $\mathrm{e}^{\pm\mathrm{i}\pi/4}$ in the expressions of the supercharges are introduced to simplify their representation in terms of creation and annihilation operators, see appendix 2.5.4.

We can lower the indices by using the skew-symmetric tensor, and in what follows we find it sometimes convenient to lower the first index and use the following charges:

$$\mathbb{L}_a{}^b = \epsilon_{ac}\mathbb{L}^{cb}, \qquad \mathbb{R}_\alpha{}^\beta = \epsilon_{\alpha\gamma}\mathbb{R}^{\gamma\beta}, \qquad \mathbb{Q}_\alpha{}^b = \epsilon_{\alpha\gamma}\mathbb{Q}^{\gamma b}, \qquad \mathbb{Q}^{\dagger\alpha}_b = \epsilon^{\alpha\gamma}\mathbb{Q}^\dagger_{b\gamma}.$$

One can check that these charges satisfy the following conditions:

$$\left(\mathbb{L}_a{}^b\right)^\dagger = \mathbb{L}_b{}^a, \qquad \mathbb{L}_1{}^1 + \mathbb{L}_2{}^2 = 0, \qquad \left(\mathbb{R}_\alpha{}^\beta\right)^\dagger = \mathbb{R}_\beta{}^\alpha,$$
$$\mathbb{R}_3{}^3 + \mathbb{R}_4{}^4 = 0, \qquad \left(\mathbb{Q}_\alpha{}^b\right)^\dagger = \mathbb{Q}^{\dagger\alpha}_b.$$

We show in the following subsection that the bosonic rotation generators $\mathbb{L}_a{}^b$, $\mathbb{R}_\alpha{}^\beta$, the supersymmetry generators $\mathbb{Q}_\alpha{}^a$, $\mathbb{Q}^{\dagger\alpha}_a$, and three central elements $\mathbb{H}$, $\mathbb{C}$ and $\mathbb{C}^\dagger$ form the centrally extended $\mathfrak{su}(2|2)$ algebra which we will denote $\mathfrak{su}(2|2)_c$. The $\mathfrak{su}(2|2)_c$ algebra relations can be written in the following form:

$$\begin{aligned}
&\left[\mathbb{L}_a{}^b, \mathbb{J}_c\right] = \delta^b_c\mathbb{J}_a - \tfrac12\delta^b_a\mathbb{J}_c, \qquad \left[\mathbb{R}_\alpha{}^\beta, \mathbb{J}_\gamma\right] = \delta^\beta_\gamma\mathbb{J}_\alpha - \tfrac12\delta^\beta_\alpha\mathbb{J}_\gamma,\\
&\left[\mathbb{L}_a{}^b, \mathbb{J}^c\right] = -\delta^c_a\mathbb{J}^b + \tfrac12\delta^b_a\mathbb{J}^c, \qquad \left[\mathbb{R}_\alpha{}^\beta, \mathbb{J}^\gamma\right] = -\delta^\gamma_\alpha\mathbb{J}^\beta + \tfrac12\delta^\beta_\alpha\mathbb{J}^\gamma,\\
&\left\{\mathbb{Q}_\alpha{}^a, \mathbb{Q}^{\dagger\beta}_b\right\} = \delta^a_b\mathbb{R}_\alpha{}^\beta + \delta^\beta_\alpha\mathbb{L}_b{}^a + \tfrac12\delta^a_b\delta^\beta_\alpha\mathbb{H},\\
&\left\{\mathbb{Q}_\alpha{}^a, \mathbb{Q}_\beta{}^b\right\} = \epsilon_{\alpha\beta}\epsilon^{ab}\mathbb{C}, \qquad \left\{\mathbb{Q}^{\dagger\alpha}_a, \mathbb{Q}^{\dagger\beta}_b\right\} = \epsilon_{ab}\epsilon^{\alpha\beta}\mathbb{C}^\dagger.
\end{aligned} \tag{2.127}$$

Here the first two lines indicate how the indices c and γ of any Lie algebra generator transform under the action of $\mathbb{L}_a{}^b$ and $\mathbb{R}_\alpha{}^\beta$. Unitarity of the string sigma model requires the world-sheet light-cone Hamiltonian $\mathbb{H}$ to be Hermitian, and the supersymmetry generators $\mathbb{Q}_\alpha{}^a$ and $\mathbb{Q}^{\dagger\alpha}_a$, and the central elements $\mathbb{C}$ and $\mathbb{C}^\dagger$ to be Hermitian conjugate to each other: $\left(\mathbb{Q}_\alpha{}^a\right)^\dagger = \mathbb{Q}^{\dagger\alpha}_a$. If one gives up the Hermiticity conditions then all the generators are considered as independent.

As we argue in the following subsection, the central elements $\mathbb{C}$ and $\mathbb{C}^\dagger$ are expressed through the world-sheet momentum $p_{\rm ws} \equiv \mathbb{P}$ as follows:

$$\mathbb{C} = \frac{\rm i}{2} g(e^{\rm i\mathbb{P}} - 1)\, e^{2{\rm i}\xi}, \qquad \mathbb{C}^\dagger = -\frac{\rm i}{2} g(e^{-\rm i\mathbb{P}} - 1)\, e^{-2{\rm i}\xi}. \tag{2.128}$$

In general, the phase ξ is an arbitrary function of the central elements. Its presence reflects the obvious fact that the algebra (2.127) admits a U(1) outer automorphism: $\mathbb{Q} \to e^{{\rm i}\xi}\mathbb{Q}, \mathbb{C} \to e^{2{\rm i}\xi}\mathbb{C}$. In perturbative string theory the phase ξ vanishes, as we will see shortly, and we find it convenient to set $\xi = 0$ for any value of the string tension g. It is important to realize that the central charges $\mathbb{C}$ and $\mathbb{C}^\dagger$ vanish on the physical subspace $\mathbb{P}|\Psi\rangle = 0$ where the usual $\mathfrak{su}(2|2)$ algebra is restored.

The remaining generators $\mathbb{L}_{\dot a}{}^{\dot b}, \mathbb{R}_{\dot\alpha}{}^{\dot\beta}, \mathbb{Q}_{\dot\alpha}{}^{\dot a}, \mathbb{Q}^{\dagger\dot\alpha}_{\dot a}$ form another copy of $\mathfrak{su}(2|2)_c$ with the same three central elements $\mathbb{H}$, $\mathbb{C}$ and $\mathbb{C}^\dagger$. Thus, the manifest symmetry algebra of the light-cone $AdS_5 \times S^5$ string sigma model coincides with the sum of two copies of $\mathfrak{su}(2|2)_c$ sharing the same set of central elements. Because of the location of the generators in the charge matrix (2.123) we will often refer to the algebras generated by undotted and dotted charges as to the left and right $\mathfrak{su}(2|2)_c$ algebras, respectively.

2.4.3. Deriving the central charges. Given the complexity of the supersymmetry generators (2.115) in the light-cone gauge as well as the corresponding Poisson structure of the theory, computation of the exact classical and quantum supersymmetry algebra is difficult. Hence, simplifying perturbative methods need to be applied. The perturbative expansion of the supersymmetry generators in powers of $1/g$ or, equivalently, in the number of fields defines a particular expansion scheme. Since in the large string tension expansion one keeps $\hat{\rm P} = g\mathbb{P}$ fixed, the corresponding expansion of the central charges starts with $-\hat{\rm P}/2$, and can be seen already at the quadratic order. This expansion, however, does not allow one to determine the exact form of the central charges (2.128) because they are non-trivial functions of $1/g$. To overcome this difficulty, in this subsection we describe a 'hybrid' expansion scheme which can be used to determine the exact form of the central charges. To be precise we determine only the part of the central charges which is independent of fermionic fields. We find that this part depends solely on the piece of the world-sheet momentum which involves the bosonic fields. Since the central charges must vanish if the world-sheet momentum does, the exact form of the central charges is, therefore, unambiguously fixed by its bosonic part.

More precisely, as can be seen from (2.115) and (2.123), a dynamical supersymmetry generator has the following generic structure:

$$\mathbb{Q}_A{}^B = \int {\rm d}\sigma\, e^{{\rm i}\alpha x_-}\, \Omega(x, p, \chi; g), \tag{2.129}$$

where the parameter α in the exponent of (2.129) is equal to $\alpha = 1/2(\epsilon_A - \epsilon_B)$, and, therefore, $\alpha = 1/2$ for supercharges $\mathbb{Q}$ and $\mathring{\mathbb{Q}}$, and $\alpha = -1/2$ for supercharges $\mathbb{Q}^\dagger$ and $\mathring{\mathbb{Q}}^\dagger$. Then, the function $\Omega(x, p, \chi; g)$ is a *local* function of transversal bosonic fields and fermionic variables. It depends on g and can be expanded, quite analogously to the Hamiltonian, in power series

$$\Omega(x, p, \chi; g) = \Omega_2(x, p, \chi) + \frac{1}{g}\Omega_4(x, p, \chi) + \cdots.$$

Here $\Omega_2(x, p, \chi)$ is quadratic in fields, $\Omega_4(x, p, \chi)$ is quartic and so on. Clearly, every term in this series also admits a finite expansion in the number of fermions. In the usual perturbative expansion we would also have to expand the non-local 'vertex' $e^{{\rm i}\alpha x_-}$ in powers of $1/g$ because

$x'_- \sim -px'/g + \cdots$. In the hybrid expansion we do not expand $e^{i\alpha x_-}$ but rather treat it as a rigid object.

The complete expression for a supercharge is rather cumbersome. However, we see that the supercharges and their algebra can be studied perturbatively: first by expanding up to a given order in $1/g$ and then by truncating the resulting polynomial up to a given number of fermionic variables. Then, as was discussed above the exact form of the central charges is completely fixed by their parts which depend only on bosons. Thus, to determine these charges it is sufficient to consider the terms in $\mathbb{Q}_A{}^B$ which are linear in fermions, and compute their Poisson brackets (or anticommutators in quantum theory) keeping only terms independent of fermions. This is, however, a complicated problem because the Poisson brackets of fermions appearing in (2.4) have a highly non-trivial dependence on bosons as have been discussed in subsection 2.1.5. We have shown in subsection 2.2.3 that to have the canonical Poisson brackets one should perform a field redefinition which can be determined up to any given order in $1/g$. Taking into account the field redefinition, integrating by parts if necessary, and using the relation $x'_- \sim -px'/g + \cdots$, one can cast any supercharge (2.129) in the following symbolic form:

$$\mathbb{Q}_A{}^B = \int d\sigma e^{i\alpha x_-}\chi\cdot\left(\Upsilon_1(x,p) + \frac{1}{g}\Upsilon_3(x,p) + \cdots\right) + \mathcal{O}(\chi^3), \tag{2.130}$$

where Υ_1 and Υ_3 are linear and cubic in bosonic fields, respectively. The explicit form of the supercharges expanded up to the leading order in $1/g$ can be found in appendix 2.5.4.

It is clear now that the bosonic part of the Poisson bracket of two supercharges is of the form

$$\begin{aligned}\{\mathbb{Q}_1,\mathbb{Q}_2\} \sim \int_{-\infty}^{\infty} d\sigma e^{i(\alpha_1+\alpha_2)x_-}\bigg(&\Upsilon_1^{(1)}(x,p)\Upsilon_1^{(2)}(x,p)\\ &+\frac{1}{g}\big(\Upsilon_1^{(1)}(x,p)\Upsilon_3^{(2)}(x,p)+\Upsilon_3^{(1)}(x,p)\Upsilon_1^{(2)}(x,p)\big)+\cdots\bigg),\end{aligned} \tag{2.131}$$

where $\mathbb{Q}_{1,2} \equiv \mathbb{Q}_{A_{1,2}}^{B_{1,2}}$. Computing the product $\Upsilon_1^{(1)}(x,p)\Upsilon_1^{(2)}(x,p)$ in the case $\alpha_1 = \alpha_2 = \pm 1/2$, we find that it is given by

$$\Upsilon_1^{(1)}(x,p)\Upsilon_1^{(2)}(x,p) \sim gx'_- + \frac{d}{d\sigma}f(x,p), \tag{2.132}$$

where $f(x,p)$ is a local function of transversal coordinates and momenta. The first term in (2.132) nicely combines with $e^{\pm ix_-}$ to give $\frac{d}{d\sigma}e^{\pm ix_-}$, and integrating this expression over σ, we obtain the sought for central charges

$$\int_{-\infty}^{\infty} d\sigma \frac{d}{d\sigma}e^{\pm ix_-} = e^{\pm ix_-(\infty)} - e^{\pm ix_-(-\infty)} = e^{\pm ix_-(-\infty)}(e^{\pm ip_{ws}} - 1), \tag{2.133}$$

where we take into account that $x_-(\infty) - x_-(-\infty) = p_{ws}$.

Making use of the explicit expressions for the supercharges from appendix 2.5.4, and identifying $x_-(-\infty) \equiv \xi$, one can easily confirm that the central charges $\mathbb{C}$ and $\mathbb{C}^\dagger$ are given by equations (2.128). Thus, the phase ξ in the central charges determines the boundary conditions for the light-cone coordinate x_-. As was mentioned above, in what follows we choose $\xi = 0$. It is worth noting however that there is another natural choice of the boundary conditions for the light-cone coordinate x_-

$$x_-(+\infty) = -x_-(-\infty) = \frac{p_{ws}}{2}.$$

This is the symmetric condition which treats both boundaries on equal footing, and leads to a real central charge

$$\mathbb{C} = \mathbb{C}^\dagger = -g\sin\left(\frac{p_{ws}}{2}\right). \tag{2.134}$$

Since we already obtained the expected central charges, the contribution of all the other terms in (2.131) should vanish. Indeed, the second term in (2.132) contributes to the order $1/g$ in the expansion as can be easily seen integrating by parts and using the relation $x'_- \sim -px'/g + \cdots$. Taking into account the additional contribution to the terms of order $1/g$ in (2.131), one can check that the total contribution is given by a σ-derivative of a local function of x and p, and, therefore, only contributes to terms of order $1/g^2$.

It is also not difficult to verify up to the quartic order in fields that the Poisson bracket of supercharges with $\alpha_1 = -\alpha_2$ gives the Hamiltonian and the kinematic generators in complete agreement with the centrally extended $\mathfrak{su}(2|2)$ algebra (2.127).

The next step is to show that the Hamiltonian commutes with all dynamical supercharges. As was already mentioned, this can be done order by order in perturbation theory in powers of the inverse string tension $1/g$ and in number of fermionic variables. One can demonstrate that up to the first non-trivial order $1/g$ the supercharge $\mathbb{Q}$ truncated to the terms linear in fermions indeed commutes with $\mathbb{H}$. To do this, one needs to keep in $\mathbb{H}$ all quadratic and quartic bosonic terms, and quadratic and quartic terms which are quadratic in fermions.

The computation we described above was purely classical, and one may want to know if quantizing the model could lead to some anomaly in the symmetry algebra. One can compute the symmetry algebra in the plane-wave limit where one keeps only quadratic terms in all the symmetry generators, and show that all potentially divergent terms cancel out and no quantum anomaly arises. The simplest way to do the computation is to use the form of the symmetry algebra generators in terms of the creation and annihilation operators from appendix (2.5.4).

Another quantum effect might be a modification of the functional dependence of the central charges on the string tension and the world-sheet momentum. It is believed, however, that the form (2.128) remains unmodified by quantum corrections, as it is consistent with both string (large g) and field (small g) theory computations of the dispersion relation.

Thus, we have shown that in the decompactification limit and for physical fields chosen to rapidly decrease at infinity the corresponding string model enjoys the symmetry which coincides with two copies of the centrally extended $\mathfrak{su}(2|2)$ algebra (2.127) sharing the same Hamiltonian and central charges.

2.5. Appendix

2.5.1. Giant magnon: explicit formulae. Here we unwrap some formulae from subsection 2.2.2 and specify them for the three simplest cases $a = 0, 1/2, 1$.

The density of the gauge-fixed Hamiltonian $\mathcal{H}$ appearing in (2.38) as a function of the coordinate z and the momentum p_z canonically conjugate to z is

$$
\begin{aligned}
\mathcal{H} = &-\frac{1-(1-a)z^2}{1-2a-(1-a)^2z^2} \\
&+ \frac{\sqrt{1+(1-z^2)(1-2a-(1-a)^2z^2)p_z^2}\sqrt{1-z^2+(1-2a-(1-a)^2z^2)z'^2}}{1-2a-(1-a)^2z^2}.
\end{aligned}
\tag{2.135}
$$

The density of the Hamiltonian (2.135) for the three simplest cases:

$$
\begin{aligned}
a = 0: &\qquad \mathcal{H} = -1 + \sqrt{\frac{1+z'^2}{1-z^2}}\sqrt{1+p_z^2(1-z^2)^2}, \\
a = \frac{1}{2}: &\qquad \mathcal{H} = -2 + \frac{4}{z^2} - \frac{1}{z^2}\sqrt{4(1-z^2)-z^2z'^2}\sqrt{4-p_z^2z^2(1-z^2)}, \\
a = 1: &\qquad \mathcal{H} = 1 - \sqrt{1-z^2-(z')^2}\sqrt{1-(1-z^2)p_z^2}.
\end{aligned}
$$

Solving the equation of motion for p_z that follows from the action (2.38), we determine the momentum as a function of $\dot{z}$ and z

$$p_z = \frac{\dot{z}}{\sqrt{(1-z^2)}\sqrt{(1-z^2)^2 - (1-2a-(1-a)^2 z^2)(\dot{z}^2 - (1-z^2)(z')^2)}}. \tag{2.136}$$

The momentum p_z as a function of $\dot{z}$ and z for the three simplest cases:

$$a=0: \qquad p_z = \frac{\dot{z}}{(1-z^2)\sqrt{1-z^2-\dot{z}^2+(1-z^2)z'^2}},$$

$$a=\frac{1}{2}: \qquad p_z = \frac{2\dot{z}}{\sqrt{1-z^2}\sqrt{4(1-z^2)^2+z^2(\dot{z}^2-(1-z^2)z'^2)}},$$

$$a=1: \qquad p_z = \frac{\dot{z}}{\sqrt{1-z^2}\sqrt{(1-z^2)^2+\dot{z}^2-(1-z^2)z'^2}}.$$

Substituting the solution (2.136) into the action (2.38), we obtain the action in the Lagrangian form

$$S = g\int_{-r}^{r} \mathrm{d}\sigma\,\mathrm{d}\tau\left(\frac{1-(1-a)z^2}{1-2a-(1-a)^2z^2} - \frac{\sqrt{(1-z^2)^2-(1-2a-(1-a)^2z^2)(\dot{z}^2-(1-z^2)z'^2)}}{\sqrt{1-z^2}(1-2a-(1-a)^2z^2)}\right). \tag{2.137}$$

The action (2.38) in the Lagrangian form for the three simplest cases:

$$a=0: \qquad S = g\int_{-r}^{r}\mathrm{d}\sigma\,\mathrm{d}\tau\left(1-\frac{\sqrt{1-z^2-\dot{z}^2+(1-z^2)z'^2}}{1-z^2}\right),$$

$$a=\frac{1}{2}: \qquad S = g\int_{-r}^{r}\mathrm{d}\sigma\,\mathrm{d}\tau\left(2-\frac{4}{z^2}+\frac{2\sqrt{4(1-z^2)^2+z^2(\dot{z}^2-(1-z^2)z'^2)}}{z^2\sqrt{1-z^2}}\right),$$

$$a=1: \qquad S = g\int_{-r}^{r}\mathrm{d}\sigma\,\mathrm{d}\tau\left(-1+\frac{\sqrt{(1-z^2)^2+\dot{z}^2-(1-z^2)z'^2}}{\sqrt{1-z^2}}\right).$$

Substituting the ansatz (2.39) into the action (2.137), we get the following Langrangian of the reduced model:

$$L_{\mathrm{red}} = \frac{1-(1-a)z^2}{1-2a-(1-a)^2z^2} - \frac{\sqrt{(1-z^2)^2+(1-2a-(1-a)^2z^2)(1-v^2-z^2)z'^2}}{\sqrt{1-z^2}(1-2a-(1-a)^2z^2)}.$$

The Hamiltonian of the reduced one-dimensional model is

$$H_{\mathrm{red}} = \pi_z z' - L_{\mathrm{red}} = -\frac{1-(1-a)z^2}{1-2a-(1-a)^2z^2} + \frac{(1-z^2)^{3/2}}{(1-2a-(1-a)^2z^2)\sqrt{(1-z^2)^2+(1-2a-(1-a)^2z^2)(1-v^2-z^2)z'^2}}.$$

2.5.2. Quartic Hamiltonian in two-index fields. We use equation (2.58) to find the following expressions for the density of the quartic Hamiltonian in terms of the two-index fields

$$\mathcal{H}_4 = \mathcal{H}_4^b + \mathcal{H}_4^f + \mathcal{H}_4^{bf},$$

where

$$\mathcal{H}_4^b = -2\big(Y^{a\dot{a}}Y_{a\dot{a}} - Z^{\alpha\dot{\alpha}}Z_{\alpha\dot{\alpha}}\big)\big(Y'^{b\dot{b}}Y'_{b\dot{b}} + Z'^{\beta\dot{\beta}}Z'_{\beta\dot{\beta}}\big) \tag{2.138}$$

is the bosonic Hamiltonian,

$$
\begin{aligned}
\mathcal{H}_4^f = \frac{1}{4}\big(&\eta^{\alpha\dot{a}}\eta^{\beta\dot{b}}\eta'_{\alpha\dot{a}}\eta'_{\beta\dot{b}} + \eta^{\alpha\dot{a}}\eta'^{\beta\dot{b}}\eta_{\alpha\dot{b}}\eta'_{\beta\dot{a}} + \eta^{\dagger\alpha\dot{a}}\eta^{\dagger\beta\dot{b}}\eta^{\dagger\prime}_{\alpha\dot{a}}\eta^{\dagger\prime}_{\beta\dot{b}} + \eta^{\dagger\alpha\dot{a}}\eta^{\dagger\prime\beta\dot{b}}\eta^{\dagger}_{\alpha\dot{b}}\eta^{\dagger\prime}_{\beta\dot{a}}\\
&+\eta^{\alpha\dot{a}}\eta^{\beta\dot{b}}\eta^{\dagger\prime}_{\alpha\dot{a}}\eta^{\dagger\prime}_{\beta\dot{b}} + \eta^{\dagger}_{\alpha\dot{a}}\eta^{\dagger}_{\beta\dot{b}}\eta'^{\alpha\dot{a}}\eta'^{\beta\dot{b}} - \eta^{\alpha\dot{a}}\eta_{\alpha\dot{b}}\eta^{\dagger\prime}_{\beta\dot{a}}\eta^{\dagger\prime\beta\dot{b}} - \eta^{\dagger}_{\alpha\dot{a}}\eta^{\dagger}_{\alpha\dot{b}}\eta'^{\beta\dot{a}}\eta'_{\beta\dot{b}}\\
&+\theta^{a\dot{\alpha}}\theta^{b\dot{\beta}}\theta'_{a\dot{\alpha}}\theta'_{b\dot{\beta}} + \theta^{a\dot{\alpha}}\theta'^{b\dot{\beta}}\theta_{b\dot{\alpha}}\theta'_{a\dot{\beta}} + \theta^{\dagger a\dot{\alpha}}\theta^{\dagger b\dot{\beta}}\theta^{\dagger\prime}_{a\dot{\alpha}}\theta^{\dagger\prime}_{b\dot{\beta}} + \theta^{\dagger a\dot{\alpha}}\theta^{\dagger\prime b\dot{\beta}}\theta^{\dagger}_{b\dot{\alpha}}\theta^{\dagger\prime}_{a\dot{\beta}}\\
&+\theta^{a\dot{\alpha}}\theta^{b\dot{\beta}}\theta^{\dagger\prime}_{a\dot{\alpha}}\theta^{\dagger\prime}_{b\dot{\beta}} + \theta^{\dagger}_{a\dot{\alpha}}\theta^{\dagger}_{b\dot{\beta}}\theta'^{a\dot{\alpha}}\theta'^{b\dot{\beta}} - \theta^{a\dot{\alpha}}\theta_{b\dot{\alpha}}\theta^{\dagger\prime}_{a\dot{\beta}}\theta^{\dagger\prime b\dot{\beta}} - \theta^{\dagger}_{a\dot{\alpha}}\theta^{\dagger}_{b\dot{\alpha}}\theta'^{a\dot{\beta}}\theta'_{b\dot{\beta}}\big)
\end{aligned}
\tag{2.139}
$$

is the fermionic Hamiltonian, and

$$
\begin{aligned}
\mathcal{H}_4^{bf} = {}&(Z^{\alpha\dot{\alpha}}Z_{\alpha\dot{\alpha}} - Y^{a\dot{a}}Y_{a\dot{a}})\big(\eta^{\dagger\prime}_{\beta\dot{b}}\eta'^{\beta\dot{b}} + \theta^{\dagger\prime}_{b\dot{\beta}}\theta'^{b\dot{\beta}}\big) - 4\mathrm{i}\big(\eta'_{\alpha\dot{a}}\theta'_{a\dot{\alpha}} + \eta^{\dagger\prime}_{\alpha\dot{a}}\theta^{\dagger\prime}_{a\dot{\alpha}}\big)Y^{a\dot{a}}Z^{\alpha\dot{\alpha}}\\
&-\frac{1}{2}\big(\eta^{\alpha\dot{a}}\eta^{\dagger\prime}_{\alpha\dot{a}} + \eta^{\dagger}_{\alpha\dot{a}}\eta'^{\alpha\dot{a}} + \theta^{a\dot{\alpha}}\theta^{\dagger\prime}_{a\dot{\alpha}} + \theta^{\dagger}_{a\dot{\alpha}}\theta'^{a\dot{\alpha}}\big)\big(Y^{b\dot{b}}Y'_{b\dot{b}} + Z^{\beta\dot{\beta}}Z'_{\beta\dot{\beta}}\big)\\
&+\big(\eta^{\alpha\dot{a}}\eta^{\dagger\prime}_{\alpha\dot{b}} + \eta^{\dagger}_{\alpha\dot{b}}\eta'^{\alpha\dot{a}}\big)Y_{a\dot{a}}Y'^{a\dot{b}} + \big(\theta^{a\dot{\alpha}}\theta^{\dagger\prime}_{b\dot{\alpha}} + \theta^{\dagger}_{b\dot{\alpha}}\theta'^{a\dot{\alpha}}\big)Y_{a\dot{a}}Y'^{b\dot{a}}\\
&+\big(\eta^{\beta\dot{a}}\eta^{\dagger\prime}_{\alpha\dot{a}} + \eta^{\dagger}_{\alpha\dot{a}}\eta'^{\beta\dot{a}}\big)Z^{\alpha\dot{\alpha}}Z'_{\beta\dot{\alpha}} + \big(\theta^{a\dot{\beta}}\theta^{\dagger\prime}_{a\dot{\alpha}} + \theta^{\dagger}_{a\dot{\beta}}\theta'^{a\dot{\alpha}}\big)Z^{\alpha\dot{\alpha}}Z'_{\alpha\dot{\beta}}\\
&+\frac{\mathrm{i}\kappa}{4}\Big(\big(\eta^{\alpha\dot{a}}\eta_{\alpha\dot{b}} + \eta^{\dagger\alpha\dot{a}}\eta^{\dagger}_{\alpha\dot{b}}\big)(P_{a\dot{a}}Y^{a\dot{b}})' + \big(\theta^{a\dot{\alpha}}\theta_{b\dot{\alpha}} + \theta^{\dagger a\dot{\alpha}}\theta^{\dagger}_{b\dot{\alpha}}\big)(P_{a\dot{a}}Y^{b\dot{a}})'\\
&+\big(\eta_{\beta\dot{a}}\eta^{\alpha\dot{a}} + \eta^{\dagger}_{\beta\dot{a}}\eta^{\dagger\alpha\dot{a}}\big)(P_{\alpha\dot{\alpha}}Z^{\beta\dot{\alpha}})' + \big(\theta_{a\dot{\beta}}\theta^{a\dot{\alpha}} + \theta^{\dagger}_{a\dot{\beta}}\theta^{\dagger a\dot{\alpha}}\big)(P_{\alpha\dot{\alpha}}Z^{\alpha\dot{\beta}})'\Big)
\end{aligned}
\tag{2.140}
$$

is the mixed Hamiltonian.

2.5.3. T-matrix. Here we list the full T-matrix in the uniform $a = 1/2$ light-cone gauge. To simplify the notations and for visual clarity we use the following notations:

$$
\begin{aligned}
&a^{\dagger}_{a\dot{a}}(p) \to Y_{a\dot{a}}, \qquad a^{\dagger}_{a\dot{a}}(p') \to Y'_{a\dot{a}}, \qquad a^{\dagger}_{\alpha\dot{\alpha}}(p) \to Z_{\alpha\dot{\alpha}}, \qquad a^{\dagger}_{\alpha\dot{\alpha}}(p') \to Z'_{\alpha\dot{\alpha}},\\
&a^{\dagger}_{\alpha\dot{a}}(p) \to \eta_{\alpha\dot{a}}, \qquad a^{\dagger}_{\alpha\dot{a}}(p') \to \eta'_{\alpha\dot{a}}, \qquad a^{\dagger}_{a\dot{\alpha}}(p) \to \theta_{a\dot{\alpha}}, \qquad a^{\dagger}_{a\dot{\alpha}}(p') \to \theta'_{a\dot{\alpha}},
\end{aligned}
$$

so that we have, in particular

$$
|Y_{a\dot{a}}\eta'_{\beta\dot{b}}\rangle \equiv \big|a^{\dagger}_{a\dot{a}}(p)a^{\dagger}_{\beta\dot{b}}(p')\big\rangle, \qquad |\theta_{a\dot{\alpha}}Z'_{\beta\dot{\beta}}\rangle \equiv \big|a^{\dagger}_{a\dot{\alpha}}(p)a^{\dagger}_{\beta\dot{\beta}}(p')\big\rangle.
$$

Then we introduce the rapidity θ related to the momentum p and energy ω as follows:

$$
p = \sinh\theta, \qquad \omega = \cosh\theta.
$$

Since the model is not Lorentz-invariant, the T-matrix does not depend only on the difference $\theta - \theta'$, and one may find the following identities useful:

$$
\begin{aligned}
&p\omega' - p'\omega = \sinh(\theta - \theta'), \qquad (p - p')\cosh\frac{\theta - \theta'}{2} = (\omega + \omega')\sinh\frac{\theta - \theta'}{2}\\
&\sinh\frac{\theta}{2} = \frac{1}{2}\sqrt{\omega + p} - \frac{1}{2}\sqrt{\omega - p}, \qquad \cosh\frac{\theta}{2} = \frac{1}{2}\sqrt{\omega + p} + \frac{1}{2}\sqrt{\omega - p}\\
&\sinh\frac{\theta - \theta'}{2} = \frac{1}{2}\sqrt{(\omega + p)(\omega' - p')} - \frac{1}{2}\sqrt{(\omega - p)(\omega' + p')}\\
&\cosh\frac{\theta - \theta'}{2} = \frac{1}{2}\sqrt{(\omega + p)(\omega' - p')} + \frac{1}{2}\sqrt{(\omega - p)(\omega' + p')}.
\end{aligned}
$$

The two momenta p and p' satisfy $p > p'$.

Boson–Boson

$$\mathbb{T}\cdot|Y_{a\dot a}Y'_{b\dot b}\rangle = +\frac{1}{2}\frac{(p-p')^2}{p\omega'-p'\omega}|Y_{a\dot a}Y'_{b\dot b}\rangle + \frac{pp'}{p\omega'-p'\omega}(|Y_{a\dot b}Y'_{b\dot a}\rangle + |Y_{b\dot a}Y'_{a\dot b}\rangle)$$
$$-\frac{pp'}{p\omega'-p'\omega}\sinh\frac{\theta-\theta'}{2}(\epsilon_{\dot a\dot b}\epsilon^{\dot\alpha\dot\beta}|\eta_{a\dot\alpha}\eta'_{b\dot\beta}\rangle + \epsilon_{ab}\epsilon^{\alpha\beta}|\theta_{\alpha\dot a}\theta'_{\beta\dot b}\rangle)$$
$$\mathbb{T}\cdot|Z_{\alpha\dot\alpha}Z'_{\beta\dot\beta}\rangle = -\frac{1}{2}\frac{(p-p')^2}{p\omega'-p'\omega}|Z_{\alpha\dot\alpha}Z'_{\beta\dot\beta}\rangle - \frac{pp'}{p\omega'-p'\omega}(|Z_{\alpha\dot\beta}Z'_{\beta\dot\alpha}\rangle + |Z_{\beta\dot\alpha}Z'_{\alpha\dot\beta}\rangle)$$
$$+\frac{pp'}{p\omega'-p'\omega}\sinh\frac{\theta-\theta'}{2}(\epsilon_{\dot\alpha\dot\beta}\epsilon^{\dot a\dot b}|\theta_{\alpha\dot a}\theta'_{\beta\dot b}\rangle + \epsilon_{\alpha\beta}\epsilon^{ab}|\eta_{a\dot\alpha}\eta'_{b\dot\beta}\rangle)$$
$$\mathbb{T}\cdot|Y_{a\dot a}Z'_{\alpha\dot\alpha}\rangle = -\frac{1}{2}\frac{p^2-p'^2}{p\omega'-p'\omega}|Y_{a\dot a}Z'_{\alpha\dot\alpha}\rangle + \frac{pp'}{p\omega'-p'\omega}\cosh\frac{\theta-\theta'}{2}(|\theta_{\alpha\dot a}\eta'_{a\dot\alpha}\rangle - |\eta_{a\dot\alpha}\theta'_{\alpha\dot a}\rangle)$$
$$\mathbb{T}\cdot|Z_{\alpha\dot\alpha}Y'_{a\dot a}\rangle = +\frac{1}{2}\frac{p^2-p'^2}{p\omega'-p'\omega}|Z_{\alpha\dot\alpha}Y'_{a\dot a}\rangle - \frac{pp'}{p\omega'-p'\omega}\cosh\frac{\theta-\theta'}{2}(|\eta_{a\dot\alpha}\theta'_{\alpha\dot a}\rangle - |\theta_{\alpha\dot a}\eta'_{a\dot\alpha}\rangle)$$

Fermion–Fermion

$$\mathbb{T}\cdot|\eta_{a\dot\alpha}\eta'_{b\dot\beta}\rangle = +\frac{pp'}{p\omega'-p'\omega}(|\eta_{b\dot\alpha}\eta'_{a\dot\beta}\rangle - |\eta_{a\dot\beta}\eta'_{b\dot\alpha}\rangle)$$
$$-\frac{pp'}{p\omega'-p'\omega}\sinh\frac{\theta-\theta'}{2}(\epsilon_{\dot\alpha\dot\beta}\epsilon^{\dot a\dot b}|Y_{a\dot a}Y'_{b\dot b}\rangle - \epsilon_{ab}\epsilon^{\alpha\beta}|Z_{\alpha\dot\alpha}Z'_{\beta\dot\beta}\rangle)$$
$$\mathbb{T}\cdot|\theta_{\alpha\dot a}\theta'_{\beta\dot b}\rangle = -\frac{pp'}{p\omega'-p'\omega}(|\theta_{\beta\dot a}\theta'_{\alpha\dot b}\rangle - |\theta_{\alpha\dot b}\theta'_{\beta\dot a}\rangle)$$
$$+\frac{pp'}{p\omega'-p'\omega}\sinh\frac{\theta-\theta'}{2}(\epsilon_{\dot a\dot b}\epsilon^{\dot\alpha\dot\beta}|Z_{\alpha\dot\alpha}Z'_{\beta\dot\beta}\rangle - \epsilon_{\alpha\beta}\epsilon^{ab}|Y_{a\dot a}Y'_{b\dot b}\rangle)$$
$$\mathbb{T}\cdot|\eta_{a\dot\alpha}\theta'_{\beta\dot b}\rangle = -\frac{pp'}{p\omega'-p'\omega}\cosh\frac{\theta-\theta'}{2}(|Y_{a\dot b}Z'_{\beta\dot\alpha}\rangle + |Z_{\beta\dot\alpha}Y'_{a\dot b}\rangle)$$
$$\mathbb{T}\cdot|\theta_{\alpha\dot a}\eta'_{b\dot\beta}\rangle = +\frac{pp'}{p\omega'-p'\omega}\cosh\frac{\theta-\theta'}{2}(|Z_{\alpha\dot\beta}Y'_{b\dot a}\rangle + |Y_{b\dot a}Z'_{\alpha\dot\beta}\rangle).$$

Boson–Fermion

$$\mathbb{T}\cdot|Y_{a\dot a}\eta'_{b\dot\beta}\rangle = +\frac{1}{2}\frac{(p'-p)p'}{p\omega'-p'\omega}|Y_{a\dot a}\eta'_{b\dot\beta}\rangle + \frac{pp'}{p\omega'-p'\omega}|Y_{b\dot a}\eta'_{a\dot\beta}\rangle$$
$$+\frac{pp'}{p\omega'-p'\omega}\cosh\frac{\theta-\theta'}{2}|\eta_{a\dot\beta}Y'_{b\dot a}\rangle - \frac{pp'}{p\omega'-p'\omega}\sinh\frac{\theta-\theta'}{2}\epsilon_{ab}\epsilon^{\alpha\beta}|\theta_{\alpha\dot a}Z'_{\beta\dot\beta}\rangle$$
$$\mathbb{T}\cdot|Y_{a\dot a}\theta'_{\beta\dot b}\rangle = +\frac{1}{2}\frac{(p'-p)p'}{p\omega'-p'\omega}|Y_{a\dot a}\theta'_{\beta\dot b}\rangle + \frac{pp'}{p\omega'-p'\omega}|Y_{a\dot b}\theta'_{\beta\dot a}\rangle$$
$$+\frac{pp'}{p\omega'-p'\omega}\cosh\frac{\theta-\theta'}{2}|\theta_{\beta\dot a}Y'_{a\dot b}\rangle + \frac{pp'}{p\omega'-p'\omega}\sinh\frac{\theta-\theta'}{2}\epsilon_{\dot a\dot b}\epsilon^{\dot\alpha\dot\beta}|\eta_{a\dot\alpha}Z'_{\beta\dot\beta}\rangle$$
$$\mathbb{T}\cdot|\eta_{a\dot\alpha}Y'_{b\dot b}\rangle = +\frac{1}{2}\frac{(p-p')p}{p\omega'-p'\omega}|\eta_{a\dot\alpha}Y'_{b\dot b}\rangle + \frac{pp'}{p\omega'-p'\omega}|\eta_{b\dot\alpha}Y'_{a\dot b}\rangle$$
$$+\frac{pp'}{p\omega'-p'\omega}\cosh\frac{\theta-\theta'}{2}|Y_{a\dot b}\eta'_{b\dot\alpha}\rangle + \frac{pp'}{p\omega'-p'\omega}\sinh\frac{\theta-\theta'}{2}\epsilon_{ab}\epsilon^{\alpha\beta}|Z_{\alpha\dot\alpha}\theta'_{\beta\dot b}\rangle$$
$$\mathbb{T}\cdot|\theta_{\alpha\dot a}Y'_{b\dot b}\rangle = +\frac{1}{2}\frac{(p-p')p}{p\omega'-p'\omega}|\theta_{\alpha\dot a}Y'_{b\dot b}\rangle + \frac{pp'}{p\omega'-p'\omega}|\theta_{\alpha\dot b}Y'_{b\dot a}\rangle$$

$$
\begin{aligned}
&\quad + \frac{pp'}{p\omega' - p'\omega}\cosh\frac{\theta-\theta'}{2}|Y_{b\dot{a}}\theta'_{\alpha\dot{b}}\rangle - \frac{pp'}{p\omega' - p'\omega}\sinh\frac{\theta-\theta'}{2}\epsilon_{\dot{a}\dot{b}}\epsilon^{\dot{\alpha}\dot{\beta}}|Z_{\alpha\dot{\alpha}}\eta'_{b\dot{\beta}}\rangle\\
\mathbb{T}\cdot|Z_{\alpha\dot{\alpha}}\eta'_{b\dot{\beta}}\rangle &= -\frac{1}{2}\frac{(p'-p)p'}{p\omega' - p'\omega}|Z_{\alpha\dot{\alpha}}\eta'_{b\dot{\beta}}\rangle - \frac{pp'}{p\omega' - p'\omega}|Z_{\alpha\dot{\beta}}\eta'_{b\dot{\alpha}}\rangle\\
&\quad - \frac{pp'}{p\omega' - p'\omega}\cosh\frac{\theta-\theta'}{2}|\eta_{b\dot{\alpha}}Z'_{\alpha\dot{\beta}}\rangle - \frac{pp'}{p\omega' - p'\omega}\sinh\frac{\theta-\theta'}{2}\epsilon_{\dot{\alpha}\dot{\beta}}\epsilon^{\dot{a}\dot{b}}|\theta_{\alpha\dot{a}}Y'_{b\dot{b}}\rangle\\
\mathbb{T}\cdot|Z_{\alpha\dot{\alpha}}\theta'_{\beta\dot{b}}\rangle &= -\frac{1}{2}\frac{(p'-p)p'}{p\omega' - p'\omega}|Z_{\alpha\dot{\alpha}}\theta'_{\beta\dot{b}}\rangle - \frac{pp'}{p\omega' - p'\omega}|Z_{\beta\dot{\alpha}}\theta'_{\alpha\dot{b}}\rangle\\
&\quad - \frac{pp'}{p\omega' - p'\omega}\cosh\frac{\theta-\theta'}{2}|\theta_{\alpha\dot{b}}Z'_{\beta\dot{\alpha}}\rangle + \frac{pp'}{p\omega' - p'\omega}\sinh\frac{\theta-\theta'}{2}\epsilon_{\alpha\beta}\epsilon^{ab}|\eta_{a\dot{\alpha}}Y'_{b\dot{b}}\rangle\\
\mathbb{T}\cdot|\eta_{a\dot{\alpha}}Z'_{\beta\dot{\beta}}\rangle &= -\frac{1}{2}\frac{(p-p')p}{p\omega' - p'\omega}|\eta_{a\dot{\alpha}}Z'_{\beta\dot{\beta}}\rangle - \frac{pp'}{p\omega' - p'\omega}|\eta_{a\dot{\beta}}Z'_{\beta\dot{\alpha}}\rangle\\
&\quad - \frac{pp'}{p\omega' - p'\omega}\cosh\frac{\theta-\theta'}{2}|Z_{\beta\dot{\alpha}}\eta'_{a\dot{\beta}}\rangle + \frac{pp'}{p\omega' - p'\omega}\sinh\frac{\theta-\theta'}{2}\epsilon_{\dot{\alpha}\dot{\beta}}\epsilon^{\dot{a}\dot{b}}|Y_{a\dot{a}}\theta'_{\beta\dot{b}}\rangle\\
\mathbb{T}\cdot|\theta_{\alpha\dot{a}}Z'_{\beta\dot{\beta}}\rangle &= -\frac{1}{2}\frac{(p-p')p}{p\omega' - p'\omega}|\theta_{\alpha\dot{a}}Z'_{\beta\dot{\beta}}\rangle - \frac{pp'}{p\omega' - p'\omega}|\theta_{\beta\dot{a}}Z'_{\alpha\dot{\beta}}\rangle\\
&\quad - \frac{pp'}{p\omega' - p'\omega}\cosh\frac{\theta-\theta'}{2}|Z_{\alpha\dot{\beta}}\theta'_{\beta\dot{a}}\rangle - \frac{pp'}{p\omega' - p'\omega}\sinh\frac{\theta-\theta'}{2}\epsilon_{\alpha\beta}\epsilon^{ab}|Y_{a\dot{a}}\eta'_{b\dot{\beta}}\rangle
\end{aligned}
$$

2.5.4. Symmetry algebra generators. The generators of the centrally extended $\mathfrak{su}(2|2) \oplus \mathfrak{su}(2|2)$ symmetry algebra up to quadratic order in the fields are given by the following expressions:

$$
\begin{aligned}
\mathbb{L}^{ab} &= \int d\sigma\left[\frac{i}{2}(\epsilon^{ac}P_{c\dot{c}}Y^{b\dot{c}} + \epsilon^{bc}P_{c\dot{c}}Y^{a\dot{c}}) - \frac{1}{4}\big(\epsilon^{ac}\theta^{\dagger}_{c\dot{\gamma}}\theta^{b\dot{\gamma}} + \epsilon^{bc}\theta^{\dagger}_{c\dot{\gamma}}\theta^{a\dot{\gamma}}\big)\right],\\
\mathbb{R}^{\alpha\beta} &= \int d\sigma\left[\frac{i}{2}(\epsilon^{\alpha\gamma}P_{\gamma\dot{\gamma}}Z^{\beta\dot{\gamma}} + \epsilon^{\beta\gamma}P_{\gamma\dot{\gamma}}Z^{\alpha\dot{\gamma}}) - \frac{1}{4}\big(\epsilon^{\alpha\gamma}\eta^{\dagger}_{\gamma\dot{c}}\eta^{\beta\dot{c}} + \epsilon^{\beta\gamma}\eta^{\dagger}_{\gamma\dot{c}}\eta^{\alpha\dot{c}}\big)\right],\\
\mathbb{Q}^{\alpha b} &= e^{-i\pi/4}\int d\sigma\,\frac{1}{2}\,e^{\frac{i}{2}x_-}\big(-i\epsilon^{\alpha\gamma}P^{b\dot{c}}\eta^{\dagger}_{\gamma\dot{c}} - 2\epsilon^{\alpha\gamma}Y^{b\dot{c}}\eta^{\dagger}_{\gamma\dot{c}} - 2\epsilon_{\dot{c}\dot{d}}Y^{b\dot{c}}\eta'^{\alpha\dot{d}}\\
&\quad - \epsilon_{\dot{\beta}\dot{\gamma}}P^{\alpha\dot{\beta}}\theta^{b\dot{\gamma}} - 2i\epsilon_{\dot{\gamma}\dot{\rho}}Z^{\alpha\dot{\gamma}}\theta^{b\dot{\rho}} - 2i\epsilon^{bc}Z^{\alpha\dot{\gamma}}\theta^{\dagger\prime}_{c\dot{\gamma}}\big),\\
\mathbb{Q}^{\dagger}_{b\alpha} &= e^{i\pi/4}\int d\sigma\,\frac{1}{2}\,e^{-\frac{i}{2}x_-}\big(i\epsilon_{\alpha\gamma}P_{b\dot{c}}\eta^{\gamma\dot{c}} - 2\epsilon_{\alpha\gamma}Y_{b\dot{c}}\eta^{\gamma\dot{c}} - 2\epsilon^{\dot{c}\dot{d}}Y_{b\dot{c}}\eta^{\dagger\prime}_{\alpha\dot{d}}\\
&\quad - \epsilon^{\dot{\beta}\dot{\gamma}}P_{\alpha\dot{\beta}}\theta^{\dagger}_{b\dot{\gamma}} + 2i\epsilon^{\dot{\gamma}\dot{\rho}}Z_{\alpha\dot{\gamma}}\theta^{\dagger}_{b\dot{\rho}} + 2i\epsilon_{bc}Z^{\alpha\dot{\gamma}}\theta'^{c\dot{\gamma}}\big),\\
\mathbb{L}^{\dot{a}\dot{b}} &= \int d\sigma\left[\frac{i}{2}\big(\epsilon^{\dot{a}\dot{c}}P_{c\dot{c}}Y^{c\dot{b}} + \epsilon^{\dot{b}\dot{c}}P_{c\dot{c}}Y^{c\dot{a}}\big) - \frac{1}{4}\big(\epsilon^{\dot{a}\dot{c}}\eta^{\dagger}_{\gamma\dot{c}}\eta^{\gamma\dot{b}} + \epsilon^{\dot{b}\dot{c}}\eta^{\dagger}_{\gamma\dot{c}}\eta^{\gamma\dot{a}}\big)\right],\\
\mathbb{R}^{\dot{\alpha}\dot{\beta}} &= \int d\sigma\left[\frac{i}{2}\big(\epsilon^{\dot{\alpha}\dot{\gamma}}P_{\gamma\dot{\gamma}}Z^{\gamma\dot{\beta}} + \epsilon^{\dot{\beta}\dot{\gamma}}P_{\gamma\dot{\gamma}}Z^{\gamma\dot{\alpha}}\big) - \frac{1}{4}\big(\epsilon^{\dot{\alpha}\dot{\gamma}}\theta^{\dagger}_{c\dot{\gamma}}\theta^{c\dot{\beta}} + \epsilon^{\dot{\beta}\dot{\gamma}}\theta^{\dagger}_{c\dot{\alpha}}\theta^{c\dot{\beta}}\big)\right],\\
\mathbb{Q}^{\dot{\alpha}\dot{b}} &= e^{-i\pi/4}\int d\sigma\,\frac{1}{2}\,e^{\frac{i}{2}x_-}\big(-i\epsilon^{\dot{\alpha}\dot{\gamma}}P^{c\dot{b}}\theta^{\dagger}_{c\dot{\gamma}} - 2\epsilon^{\dot{\alpha}\dot{\gamma}}Y^{c\dot{b}}\theta^{\dagger}_{c\dot{\gamma}} - 2\epsilon_{cd}Y^{c\dot{b}}\theta'^{d\dot{\alpha}}\\
&\quad + \epsilon_{\beta\gamma}P^{\beta\dot{\alpha}}\eta^{\gamma\dot{b}} + 2i\epsilon_{\gamma\rho}Z^{\gamma\dot{\alpha}}\eta^{\rho\dot{b}} + 2i\epsilon^{\dot{b}\dot{c}}Z^{\gamma\dot{\alpha}}\eta^{\dagger\prime}_{\gamma\dot{c}}\big),\\
\mathbb{Q}^{\dagger}_{\dot{b}\dot{\alpha}} &= e^{i\pi/4}\int d\sigma\,\frac{1}{2}\,e^{-\frac{i}{2}x_-}\big(i\epsilon_{\dot{\alpha}\dot{\gamma}}P_{c\dot{b}}\theta^{c\dot{\gamma}} - 2\epsilon_{\dot{\alpha}\dot{\gamma}}Y_{c\dot{b}}\theta^{d\dot{\gamma}} - 2\epsilon_{\dot{b}\dot{c}}Y^{c\dot{c}}\theta^{\dagger\prime}_{c\dot{\alpha}}\\
&\quad + \epsilon^{\beta\gamma}P_{\beta\dot{\alpha}}\eta^{\dagger}_{\gamma\dot{b}} - 2i\epsilon_{\dot{\alpha}\dot{\gamma}}Z^{\beta\dot{\gamma}}\eta^{\dagger}_{\beta\dot{b}} - 2i\epsilon_{\dot{b}\dot{c}}Z_{\gamma\dot{\alpha}}\eta'^{\gamma\dot{c}}\big),
\end{aligned}
$$

and the Hamiltonian $\mathbb{H}$ and the world-sheet momentum $\mathbb{P}$ up to quadratic order in the fields are given by

$$\mathbb{H}_2 = \int d\sigma \bigg(\frac{1}{4} P_{a\dot{a}} P^{a\dot{a}} + Y_{a\dot{a}} Y^{a\dot{a}} + Y'_{a\dot{a}} Y'^{a\dot{a}} + \frac{1}{4} P_{\alpha\dot{\alpha}} P^{\alpha\dot{\alpha}} + Z_{\alpha\dot{\alpha}} Z^{\alpha\dot{\alpha}} + Z'_{\alpha\dot{\alpha}} Z'^{\alpha\dot{\alpha}}$$
$$+ \eta^\dagger_{\alpha\dot{a}} \eta^{\alpha\dot{a}} + \frac{\kappa}{2} \eta^{\alpha\dot{a}} \eta'_{\alpha\dot{a}} - \frac{\kappa}{2} \eta^{\dagger\alpha\dot{a}} \eta'^\dagger_{\alpha\dot{a}} + \theta^\dagger_{a\dot{\alpha}} \theta^{a\dot{\alpha}} + \frac{\kappa}{2} \theta^{a\dot{\alpha}} \theta'_{a\dot{\alpha}} - \frac{\kappa}{2} \theta^{\dagger a\dot{\alpha}} \theta'^\dagger_{a\dot{\alpha}} \bigg),$$

$$\mathbb{P} = \frac{\hat{\mathrm{P}}}{g} = -\frac{1}{g} \int d\sigma \big(P_{a\dot{a}} Y'^{a\dot{a}} + P_{\alpha\dot{\alpha}} Z'^{\alpha\dot{\alpha}} + i \theta^\dagger_{\alpha\dot{a}} \theta'^{\alpha\dot{a}} + i \eta^\dagger_{a\dot{\alpha}} \eta'^{a\dot{\alpha}} \big).$$

Lowering the first (or raising the second) index and omitting $e^{\pm i x_-/2}$, one gets the following expressions for these charges in terms of the creation and annihilation operators:

$$\mathbb{L}_a{}^b = \int dp \sum_{\dot{M}} \frac{1}{2} \big(a^\dagger_{a\dot{M}} a^{b\dot{M}} - \epsilon_{ad} \epsilon^{bc} a^\dagger_{c\dot{M}} a^{d\dot{M}} \big),$$
$$\mathbb{R}_\alpha{}^\beta = \int dp \sum_{\dot{M}} \frac{1}{2} \big(a^\dagger_{\alpha\dot{M}} a^{\beta\dot{M}} - \epsilon_{\alpha\rho} \epsilon^{\beta\gamma} a^\dagger_{\gamma\dot{M}} a^{\rho\dot{M}} \big),$$
$$\mathbb{Q}_\alpha{}^b = \int dp \sum_{\dot{M}} \big(f_p a^\dagger_{\alpha\dot{M}} a^{b\dot{M}} - h_p \epsilon_{\alpha\gamma} \epsilon^{bc} a^\dagger_{c\dot{M}} a^{\gamma\dot{M}} \big),$$
$$\mathbb{Q}^{\dagger\alpha}_b = \int dp \sum_{\dot{M}} \big(f_p a^\dagger_{b\dot{M}} a^{\alpha\dot{M}} - h_p \epsilon^{\alpha\gamma} \epsilon_{bc} a^\dagger_{\gamma\dot{M}} a^{c\dot{M}} \big),$$
$$\mathbb{L}_{\dot{a}}{}^{\dot{b}} = \int dp \sum_M \frac{1}{2} \big(a^\dagger_{M\dot{a}} a^{M\dot{b}} - \epsilon_{\dot{a}\dot{d}} \epsilon^{\dot{b}\dot{c}} a^\dagger_{M\dot{c}} a^{M\dot{d}} \big),$$
$$\mathbb{R}_{\dot{\alpha}}{}^{\dot{\beta}} = \int dp \sum_M \frac{1}{2} \big(a^\dagger_{M\dot{\alpha}} a^{M\dot{\beta}} - \epsilon_{\dot{\alpha}\dot{\rho}} \epsilon^{\dot{\beta}\dot{\gamma}} a^\dagger_{M\dot{\gamma}} a^{M\dot{\rho}} \big),$$
$$\mathbb{Q}_{\dot{\alpha}}{}^{\dot{b}} = \int dp \sum_M (-1)^{\epsilon_M} \big(f_p a^\dagger_{M\dot{\alpha}} a^{M\dot{b}} - h_p \epsilon_{\dot{\alpha}\dot{\gamma}} \epsilon^{\dot{b}\dot{c}} a^\dagger_{M\dot{c}} a^{M\dot{\gamma}} \big),$$
$$\mathbb{Q}^{\dagger\dot{\alpha}}_{\dot{b}} = \int dp \sum_M (-1)^{\epsilon_M} \big(f_p a^\dagger_{M\dot{b}} a^{M\dot{\alpha}} - h_p \epsilon^{\dot{\alpha}\dot{\gamma}} \epsilon_{\dot{b}\dot{c}} a^\dagger_{M\dot{\gamma}} a^{M\dot{c}} \big),$$
$$\mathbb{H}_2 = \int dp \sum_{M,\dot{M}} \omega_p a^\dagger_{M\dot{M}} a^{M\dot{M}}, \qquad \mathbb{P} = \frac{\hat{\mathrm{P}}}{g} = \frac{1}{g} \int dp \sum_{M,\dot{M}} p a^\dagger_{M\dot{M}} a^{M\dot{M}}.$$

2.5.5. Poisson brackets and the moment map. The group PSU(2, 2|4) acts on the coset space (1.1) by multiplication of a coset element by a group element from the left. Fixing the light-cone gauge and solving the Virasoro constraints, we obtain a well-defined symplectic structure ω (the inverse of the Poisson bracket) for physical fields. Therefore, now we are able to study the Poisson algebra of the Noether charges corresponding to infinitesimal global symmetry transformations generated by the Lie algebra $\mathfrak{psu}(2,2|4)$. In the first place we are interested in those charges which leave the gauge-fixed Hamiltonian and, as a consequence, the symplectic structure of the theory invariant; the corresponding subspace in $\mathfrak{psu}(2,2|4)$ will be called $\mathcal{J}$.

Since the symplectic form ω remains invariant under the action of $\mathcal{J}$, to every element $\mathcal{M} \in \mathcal{J}$ one can associate a locally Hamiltonian phase flow $\xi_\mathcal{M}$ with the Hamiltonian function being the Noether charge $\mathrm{Q}_\mathcal{M}$

$$\omega(\xi_\mathcal{M}, \ldots) + d\,\mathrm{Q}_\mathcal{M} = 0. \tag{2.141}$$

Identifying $\mathfrak{psu}(2,2|4)$ with its dual space, $\mathfrak{psu}(2,2|4)^*$, by using the supertrace operation, we can treat the matrix Q as the *moment map* which maps the phase space (x,p,χ) into the dual space to the Lie algebra

$$\mathrm{Q} : (x,p,\chi) \to \mathfrak{psu}(2,2|4)^*$$

and it allows one to associate to any element $\mathcal{M}$ of $\mathfrak{psu}(2,2|4)$ a function $\mathrm{Q}_{\mathcal{M}}$ on the phase space. This linear mapping from the Lie algebra into the space of functions on the phase space is given by equation (2.119). The function $\mathrm{Q}_{\mathcal{M}}$ is a Hamiltonian function, i.e. it obeys equation (2.141), only if $\mathcal{M} \in \mathcal{J}$. Although the elements of $\mathfrak{psu}(2,2|4)$ which do not belong to $\mathcal{J}$ are symmetries of the gauge-fixed action, they leave neither the Hamiltonian nor the symplectic structure invariant.

As is well known, equation (2.141) implies the following general formula for the Poisson bracket of the Noether charges $\mathrm{Q}_{\mathcal{M}}$:

$$\{\mathrm{Q}_{\mathcal{M}_1}, \mathrm{Q}_{\mathcal{M}_2}\} = (-1)^{\epsilon_{\mathcal{M}_1}\epsilon_{\mathcal{M}_2}}\,\mathrm{str}(\mathrm{Q}[\mathcal{M}_1,\mathcal{M}_2]) + \mathtt{C}(\mathcal{M}_1,\mathcal{M}_2), \tag{2.142}$$

where $\mathcal{M}_{1,2} \in \mathcal{J}$. Here $\epsilon_{\mathcal{M}}$ is the parity of a supermatrix $\mathcal{M}$ and $[\mathcal{M}_1,\mathcal{M}_2]$ is the graded commutator, i.e. it is the anti-commutator if both $\mathcal{M}_1$ and $\mathcal{M}_2$ are odd matrices, and the commutator if at least one of them is even. The first term in the right-hand side of equation (2.142) reflects the fact that the Poisson bracket of the Noether charges $\mathrm{Q}_{\mathcal{M}_1}$ and $\mathrm{Q}_{\mathcal{M}_2}$ gives a charge corresponding to the commutator $[\mathcal{M}_1,\mathcal{M}_2]$. The normalization prefactor $(-1)^{\epsilon_{\mathcal{M}_1}\epsilon_{\mathcal{M}_2}}$ is of no great importance and it is related to our specific choice of normalizing the even elements with respect to the odd ones inside the matrix Q. The quantity $\mathtt{C}(\mathcal{M}_1,\mathcal{M}_2)$ in the right-hand side of equation (2.142) is the central extension, i.e. a bilinear graded skew-symmetric form on the Lie algebra $\mathcal{J}$. It Poisson-commutes with all $\mathrm{Q}_{\mathcal{M}}$, $\mathcal{M} \in \mathcal{J}$. The Jacobi identity for the bracket (2.142) implies that $\mathtt{C}(\mathcal{M}_1,\mathcal{M}_2)$ is a two-dimensional cocycle of the Lie algebra $\mathcal{J}$. For simple Lie algebras such a cocycle necessarily vanishes, while for super Lie algebras it is generally not the case. Since we consider a finite-dimensional super Lie algebra the central extension vanishes if the element $\mathcal{M}$ is bosonic: $\mathtt{C}(\mathcal{M},\ldots) = 0$.

Some comments are necessary here. As we already mentioned, the standard feature of the light-cone closed string theory is the presence of the level-matching constraint $p_{\mathrm{ws}} = 0$. In the off-shell theory we rather keep p_{ws} non-vanishing. The light-cone Hamiltonian commutes with p_{ws}: $\{H, p_{\mathrm{ws}}\} = 0$, i.e. p_{ws} is an integral of motion. The Poisson bracket (2.142) with the vanishing central term is valid on-shell and it is the off-shell theory where one could expect the appearance of a non-trivial central extension. Below we determine a general form of the central extension based on symmetry arguments only. The explicit evaluation of the Poisson brackets which justifies formula (2.142) was discussed in the main text.

Let us note that formula (2.142) makes it easy to reobtain our results on the structure of $\mathcal{J}$. Indeed, from equation (2.142) we find that the invariance subalgebra $\mathcal{J} \subset \mathfrak{psu}(2,2|4)$ of the Hamiltonian is determined by the condition

$$\{H, \mathrm{Q}_{\mathcal{M}}\} = \mathrm{str}(\mathrm{Q}[\Sigma_+,\mathcal{M}]) = 0.$$

Thus, $\mathcal{J}$ is the stabilizer of the element Σ_+ in $\mathfrak{psu}(2,2|4)$

$$[\Sigma_+,\mathcal{M}] = 0, \qquad \mathcal{M} \in \mathcal{J}.$$

Obviously, $\mathcal{J}$ coincides with the red–blue submatrix of $\mathcal{M}$ in figure 3. Thus, for P_+ being finite[20] we would obtain the following vector space decomposition of $\mathcal{J}$:

$$\mathcal{J} = \mathfrak{psu}(2|2) \oplus \mathfrak{psu}(2|2) \oplus \Sigma_+ \oplus \Sigma_-.$$

[20] For P_+ finite the subalgebra which leaves invariant both H and P_+ coincides with the even subalgebra $\mathcal{J}_{\mathrm{even}}$ of $\mathcal{J}$. In fact $\mathcal{J}_{\mathrm{even}}$ is nothing else but the algebra $\mathfrak{C}$ defined in (1.127). Indeed, according to equations (2.120) and (2.121), $\mathcal{J}_{\mathrm{even}}$ arises as the simultaneous solution the two equations, $[\Sigma_+,\mathcal{M}] = 0$ and $[\Sigma_-,\mathcal{M}] = 0$ or, in other words, it is the centralizer of $\Lambda(t,\phi)$ given by equation (1.121). Together with $\Sigma_\pm$ the algebra $\mathfrak{C}$ comprises the red and blue *diagonal* blocks in figure 3.

The rank of the latter subalgebra is six and it coincides with that of $\mathfrak{psu}(2,2|4)$. In the case of infinite P_+ the last generator decouples.

Now we are ready to determine the general form of the central term in equation (2.142). Denote by $\mathcal{J}_{\rm even} \subset \mathcal{J}$ the even (bosonic) subalgebra of $\mathcal{J}$. It is represented by the red and blue diagonal blocks in figure 3. Let $G_{\rm even}$ be the corresponding group. The adjoint action of $G_{\rm even}$ preserves the $\mathbb{Z}_2$-grading of $\mathcal{J}$. Obviously, if we perform the transformation

$$\mathrm{Q} \to g\,\mathrm{Q}g^{-1}, \qquad \mathcal{M} \to g^{-1}\mathcal{M}g$$

with an element $g \in G_{\rm even}$ the charge $\mathrm{Q}_{\mathcal{M}}$ remains invariant. This transformation leaves the lhs of the bracket (2.142) invariant. As a consequence, the central term must satisfy the following invariance condition:

$$\mathtt{C}(g\mathcal{M}_1g^{-1}, g\mathcal{M}_2g^{-1}) = \mathtt{C}(\mathcal{M}_1, \mathcal{M}_2). \tag{2.143}$$

It is not difficult to find a general expression for a bilinear graded skew-symmetric form on $\mathcal{J}$ which satisfies this condition. It is given by

$$\mathtt{C}(\mathcal{M}_1, \mathcal{M}_2) = \mathrm{str}\big(\big(\varrho\mathcal{M}_1\varrho\mathcal{M}_2^t + (-1)^{\epsilon_{\mathcal{M}_1}\epsilon_{\mathcal{M}_2}}\varrho\mathcal{M}_2\varrho\mathcal{M}_1^t\big)\Phi\big). \tag{2.144}$$

Here

$$\Phi = -\frac{1}{2}\begin{pmatrix} c_3\mathbb{1}_2 & 0 & 0 & 0 \\ 0 & c_1\mathbb{1}_2 & 0 & 0 \\ 0 & 0 & c_4\mathbb{1}_2 & 0 \\ 0 & 0 & 0 & c_2\mathbb{1}_2 \end{pmatrix}, \tag{2.145}$$

where $\mathbb{1}_2$ is the two-dimensional identity matrix and

$$\varrho = \begin{pmatrix} \epsilon & 0 & 0 & 0 \\ 0 & \epsilon & 0 & 0 \\ 0 & 0 & \epsilon & 0 \\ 0 & 0 & 0 & \epsilon \end{pmatrix},$$

where ϵ is defined in equation (1.131). Note that ϱ is essentially the charge conjugation matrix. Condition (2.143) follows from the form of the matrix Φ and the equation

$$\mathcal{J}_{\rm even}^t\varrho + \varrho\mathcal{J}_{\rm even} = 0.$$

The coefficients c_i, $i = 1, \ldots, 4$ can depend on the physical fields and they are central with respect to the action of $\mathcal{J}$

$$\{c_i, \mathrm{Q}_{\mathcal{M}}\} = 0, \mathcal{M} \in \mathcal{J}.$$

By using equation (2.144) one can check that the cocycle condition for $\mathtt{C}(\mathcal{M}_1, \mathcal{M}_2)$ is trivially satisfied. In accordance with our assumptions, $\mathtt{C}(\mathcal{M}_1, \mathcal{M}_2)$ does not vanish only if both $\mathcal{M}_1$ and $\mathcal{M}_2$ are odd.

Taking into account that $\mathcal{J}$ contains two identical subalgebras $\mathfrak{psu}(2|2)$ we can put $c_1 = c_3$ and $c_2 = c_4$. Thus, general symmetry arguments fix the form of the central extension up to two central functions c_1 and c_2. Since we consider the algebra $\mathfrak{psu}(2|2)$, which is the real form of $\mathfrak{psl}(2|2)$, the conjugation rule implies that $c_1 = -c_2^*$.

2.6. Bibliographic remarks

The phase-space light-cone gauge for strings in flat space was introduced in [83]. It can be generalized to strings moving in a curved background with at least one time and one space isometry directions. If one chooses the time and space isometries from the AdS part of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ background one gets the light-cone gauge by [84]. The uniform light-cone gauge we discuss was introduced in [13, 85, 86], and belongs to the class of gauges used to study the dynamics of spinning strings in $AdS_5 \times S^5$ [87, 88].

The BMN limit was introduced in the paper by Berenstein, Maldacena and Nastase [4]. In this limit the string sigma model on $AdS_5 \times S^5$ reduces to that describing strings in the plane-wave background [89, 90]. In the light-cone gauge this string sigma model is a free theory of massive bosons and fermions, and it has been analyzed in [91, 92]. The $1/P_+$ corrections to the energy of string states were studied in [93–97, 14]. As was shown in [13], the $a = 0$ uniform gauge is in fact a non-perturbative version of the perturbative light-cone gauge used in [94–97].

The first-order formalism for the $AdS_5 \times S^5$ superstring model, the full gauge-fixed Lagrangian and its expansion up to quartic order were found in [14]; we follow this work very closely in section 3. The reader might consult [14] for more details and missing derivations.

The decompactification limit was discussed in many papers, see e.g. [98–101]. One-soliton solutions were identified with spin chain magnons and named 'giant magnons' in [101]. The giant magnon solution was found in [101] by employing the conformal gauge. The derivation of the light-cone gauge giant magnon solution and its dispersion relation in subsection 2.2.2 follows closely [86].

The two-index notation for physical fields of the light-cone model was introduced in [81]. Our fields, however, differ from those in [81] by various factors. As a result, our expressions for the supercharges in appendix 2.5.4 are slightly different from those in [81]. Nevertheless, the T-matrix coincides with that computed there. The formulae for the T-matrix in subsection 2.3 and in appendix 2.5.3 are taken from [81].

Formula (2.115) for the $\mathfrak{psu}(2,2|4)$ charges was obtained in [14]. The centrally extended $\mathfrak{su}(2|2)$ algebra was derived by using the hybrid expansion scheme in [15]. Given that the central charges retain their functional form in quantum theory, the algebra allows one to uniquely determine the dispersion relation. The dispersion relation implied by equation (2.128) has been verified in field theory up to the fourth order [36] and in string theory up to the second order [33]. The centrally extended $\mathfrak{su}(2|2)$ algebra coincides with the one previously suggested in the gauge theory spin chain context in [16]. There is however no gauge theory derivation of the centrally extended algebra.

For the notion of the moment map and related issues discussed in appendix 2.5.2 we refer to [102–104].

3. World-sheet *S*-matrix

In the previous sections we have demonstrated integrability of the classical string sigma model and developed the semi-classical quantization scheme based on the large tension expansion. The scattering matrix of world-sheet excitations has been computed in the Born approximation. We have also shown that in the off-shell string theory the symmetry algebra of the light-cone Hamiltonian coincides with two copies of the centrally extended $\mathfrak{psu}(2|2)$ superalgebra sharing the same set of central charges.

Given the current lack of non-perturbative quantization schemes, occurrence of integrability in the corresponding quantum model is much harder to establish. Because of ultra-violet and infra-red divergences arising in the process of perturbative quantization, the definition of the quantum model itself is far from obvious. At best, it should rely on finding regularization and renormalization schemes in the world-sheet theory which would allow one to uplift the classical conservation laws to the quantum level. In view of this, to make progress we will employ a 'top-to-bottom' approach. Namely, we will *assume that our model is quantum integrable* and then will derive the corresponding consequences. The

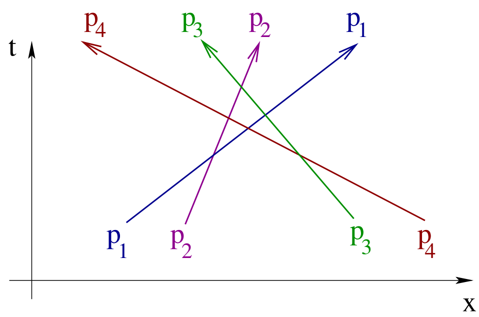

Figure 4. Factorization of the multi-particle scattering.

results obtained should obviously agree with available gauge and string perturbative data in order to make quantum integrability plausible. Moreover, in certain cases the results gathered in perturbative calculations will be essentially used to fix the structures which remain undetermined from our assumption of quantum integrability.

In the decompactification limit when the circumference of the world-sheet cylinder tends to infinity, the effective sigma model arising on the plane is massive. The massive character of a theory usually implies that interactions fall off sufficiently fast with distance, so that the concept of asymptotic states and their scattering makes sense. Under these circumstances quantum integrability can be understood as the absence of particle production and factorization of the multi-particle scattering into a sequence of two-body events.

In this section we will treat the string sigma model in the framework of the factorized scattering theory. We will show that the symmetry principles alone lead to almost complete determination of the exact world-sheet S-matrix and that the latter satisfies the standard axioms of the factorized scattering theory. Besides the centrally extended $\mathfrak{psu}(2|2)$ symmetry algebra, an important role in our treatment will be played by crossing symmetry which exchanges particles with anti-particles. Compatibility of scattering with crossing symmetry will imply a non-trivial functional equation for an overall phase of the world-sheet S-matrix; the latter can not be constrained by other known symmetries or by the requirement of factorization. We will present some physically interesting solutions to this functional equation and discuss the properties of the corresponding world-sheet S-matrix.

3.1. Elements of factorized scattering theory

Consider scattering in a two-dimensional quantum field theory that exhibits an infinite number of conservation laws (charges) $\mathbf{q}_k$, $k = 1, \ldots, \infty$, which all mutually commute. Obviously, there exists a basis of one-particle states in which these charges act diagonally

$$\mathbf{q}_k|p\rangle = q_k(p)|p\rangle.$$

If these charges are functionally independent then the corresponding scattering theory turns out to be highly constrained. First, the number of particles cannot change in the collision process; particle production is absent. Second, additivity of the conservation laws implies that

$$\sum_{j\in \text{in}} q_k(p_j) = \sum_{j\in \text{out}} q_k(p_j) \qquad \text{for any } k.$$

Thus, the set of initial momenta is preserved under collision, the particles are only allowed to exchange their individual momenta and flavors, see figure 4. In other words, scattering is elastic. Finally, an infinite tower of conservation laws implies that the multi-particle S-matrix factorizes into the product of two-particle ones.

In this section we recall the basic concepts of factorized scattering theory. First, we describe the Hilbert space of the asymptotic states as a representation carrier of the Zamolodchikov–Faddeev (ZF) algebra; the latter is a deformed algebra of creation and annihilation operators with defining relations given by the scattering matrix. Second, we derive the constraints imposed by symmetries of the Hamiltonian on the scattering matrix. Finally, we show that the physicality requirements on the S-matrix coincide with those which follow from the compatibility of the ZF algebra relations.

3.1.1. Zamolodchikov–Faddeev algebra. Let $\mathscr{J}$ be the symmetry algebra of our quantum integrable model which leaves the vacuum state $|\Omega\rangle$ invariant. Introduce a creation operator $A_i^\dagger(p)$ which creates a multiplet $\mathscr{V}$ of particles out of the vacuum with momentum p transforming in a linear irreducible representation of $\mathscr{J}$. Here index i labels various states in this multiplet (the flavor index). The Hermitian conjugate $A^i(p)$ is the vacuum annihilation operator

$$A^i(p)|\Omega\rangle = 0.$$

States in the multiplet may have different statistics and, for this reason, it is convenient to define parity ϵ_i, the latter being equal to zero or one depending on whether the value of i corresponds to a bosonic or fermionic state, respectively.

To describe the scattering process, we introduce the in-basis and the out-basis of asymptotic states as

$$|p_1, p_2, \ldots, p_n\rangle^{(in)}_{i_1,\ldots,i_n} = A^\dagger_{i_1}(p_1)\cdots A^\dagger_{i_n}(p_n)|\Omega\rangle, \qquad p_1 > p_2 > \cdots > p_n,$$
$$|p_1, p_2, \ldots, p_n\rangle^{(out)}_{i_1,\ldots,i_n} = (-1)^{\sum_{k<l}\epsilon_{i_k}\epsilon_{i_l}} A^\dagger_{i_n}(p_n)\cdots A^\dagger_{i_1}(p_1)|\Omega\rangle, \qquad p_1 > p_2 > \cdots > p_n.$$

The in and out states are the eigenstates of the Hamiltonian $\mathbb{H}$ of the model and the ordering of momenta is essential. The operators $A^\dagger(p)$ should not be confused with the fields $a^{\dagger\text{in/out}}(p), a^\dagger(p)$ introduced in section 2.3. In terms of the Heisenberg creation operators the in and out states read as

$$|p_1, p_2, \ldots, p_n\rangle^{(in)}_{i_1,\ldots,i_n} = a^{\dagger\text{in}}_{i_1}(p_1)\cdots a^{\dagger\text{in}}_{i_n}(p_n)|\Omega\rangle,$$
$$|p_1, p_2, \ldots, p_n\rangle^{(out)}_{i_1,\ldots,i_n} = a^{\dagger\text{out}}_{i_1}(p_1)\cdots a^{\dagger\text{out}}_{i_n}(p_n)|\Omega\rangle,$$

where the ordering of particle momenta is the same as in the formulae above.

The operators $A_i^\dagger$ and A^i are known as the ZF creation and annihilation operators, respectively. In contrast to $a_i^\dagger$ and a^i, these operators do not satisfy the canonical commutation relations in interacting theory. In the free field limit the ZF operators turn into $a_i^\dagger$ and a^i, which explains an extra statistics-carrying factor $(-1)^{\sum_{k<l}\epsilon_{i_k}\epsilon_{i_l}}$ in the above formula for the out-states.

In our new description of asymptotic states, scattering is understood as reordering of particles (creation operators) in the momentum space. Particles can be distinguishable, each of them carrying a definite flavor (the value of index i). Then, in the two-body collision process particles can either keep their individual momenta, which is forward scattering (transition), or exchange the latter (in the case of equal mass), which is backward scattering (reflection), see figure 5. Note that the very possibility of describing the asymptotic states and their scattering in such a fashion is due to S-matrix factorization, as it will become apparent in a moment.

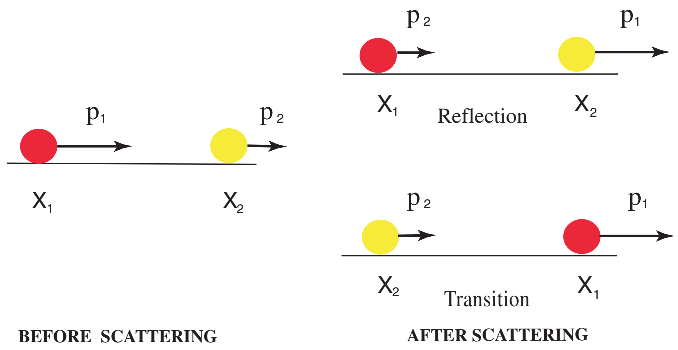

Figure 5. In the collision process particles either keep (transition) or exchange (reflection) their momenta. The S-matrix operates non-trivially in the flavor space.

According to the discussion in the previous section, the in and out states are related by the unitary S-matrix operator $\mathbb{S}$:

$$|p_1,\ldots,p_n\rangle^{(in)}_{i_1,\ldots,i_n} = \mathbb{S}\cdot|p_1,\ldots,p_n\rangle^{(out)}_{i_1,\ldots,i_n}, \tag{3.1}$$

and one can expand initial states on a basis of final states and vice versa. In particular, the two-particle in and out states are related by equation (2.85), which now takes the form

$$A^\dagger_i(p_1)A^\dagger_j(p_2)|\Omega\rangle = \mathbb{S}\cdot(-1)^{\epsilon_i\epsilon_j}A^\dagger_j(p_2)A^\dagger_i(p_1)|\Omega\rangle = \mathbb{S}^{kl}_{ij}(p_1,p_2)(-1)^{\epsilon_k\epsilon_l}A^\dagger_l(p_2)A^\dagger_k(p_1)|\Omega\rangle.$$

This formula suggests to define the new matrix elements as

$$S^{kl}_{ij}(p_1,p_2)\equiv\mathbb{S}^{kl}_{ij}(p_1,p_2)(-1)^{\epsilon_k\epsilon_l}. \tag{3.2}$$

Now, by discarding the vacuum state on both sides of the formula just above equation (3.2), we obtain the following algebra of creation operators:

$$A^\dagger_i(p_1)A^\dagger_j(p_2) = A^\dagger_l(p_2)A^\dagger_k(p_1)S^{kl}_{ij}(p_1,p_2), \tag{3.3}$$

which is usually referred to as the ZF algebra.

Before stating the consistency conditions of these algebra relations, it is convenient to rewrite (3.3) in the matrix form. To this end, we introduce rows E_i and columns E^i with all vanishing entries except the one in the ith position which is equal to the identity. The standard matrix unities are then $E_i^j = E_i\otimes E^j$ with the only non-vanishing element equal to the identity which occurs on the intersection of the ith row with the jth column. The following multiplication rules are valid $E^kE_i^j = \delta_i^kE^j$ and $E_i^jE_k = \delta_k^jE_i$ together with the product rule for the matrix unities: $E_i^jE_k^l = \delta_k^jE_i^l$. With this notation at hand we can represent the ZF creation and annihilation operators as rows and columns, respectively,

$$\mathbb{A}^\dagger = A^\dagger_iE^i,\qquad \mathbb{A} = A^iE_i, \tag{3.4}$$

while the entities (3.2) can be combined in the following matrix:

$$S(p_1,p_2) = S^{kl}_{ij}(p_1,p_2)E^i_k\otimes E^j_l \tag{3.5}$$

which is an element in $\mathrm{End}(\mathscr{V}\otimes\mathscr{V})$. Thus, in the matrix notation the relations (3.3) acquire the form

$$\mathbb{A}^\dagger_1(p_1)\mathbb{A}^\dagger_2(p_2) = \mathbb{A}^\dagger_2(p_2)\mathbb{A}^\dagger_1(p_1)S_{12}(p_1,p_2), \tag{3.6}$$

where $S_{12}\equiv S$, and we use the following convention:

$$\mathbb{A}^\dagger_1\mathbb{A}^\dagger_2 = A^\dagger_i(p_1)A^\dagger_j(p_2)E^i\otimes E^j,\qquad \mathbb{A}^\dagger_2\mathbb{A}^\dagger_1 = A^\dagger_j(p_2)A^\dagger_i(p_1)E^i\otimes E^j.$$

In what follows, if A, B, C are either columns or rows with operator entries then in the notation $A_1B_2C_3$ the subscripts 1, 2, 3 refer to the location of the columns and rows, e.g. if $A = A^i(p_3)E_i$, $B = B_i(p_1)E^i$, $C = C_i(p_2)E^i$, then $A_1B_3C_2 = A^i(p_3)B_k(p_1)C_j(p_2)E_i \otimes E^j \otimes E^k$.

This formula can be naturally supplemented by similar relations between two annihilation operators and between creation and annihilation operators, so that the complete algebra relations look like

$$\mathbb{A}_1^\dagger\mathbb{A}_2^\dagger = \mathbb{A}_2^\dagger\mathbb{A}_1^\dagger S_{12}, \qquad \mathbb{A}_1\mathbb{A}_2 = S_{12}\mathbb{A}_2\mathbb{A}_1, \qquad \mathbb{A}_1\mathbb{A}_2^\dagger = \mathbb{A}_2^\dagger S_{21}\mathbb{A}_1 + \delta_{12}. \tag{3.7}$$

Here $S_{21} = S_{ij}^{kl}(p_2, p_1)E_l^j \otimes E_k^i$, and $\delta_{12} = \delta(p_1 - p_2)E_i \otimes E^i$, where summation over repeated indices is assumed. In what follows we will need the following three matrices known as the permutation matrix P, the graded permutation P^g and the graded identity $\mathbb{1}^g$:

$$P = E_i^j \otimes E_j^i, \qquad P^g = (-1)^{\epsilon_i\epsilon_j}E_i^j \otimes E_j^i, \qquad \mathbb{1}^g = (-1)^{\epsilon_i\epsilon_j}E_i^i \otimes E_j^j. \tag{3.8}$$

The permutation matrix transforms S_{12} into S_{21}: $PS_{12}(p, p') = S_{21}(p, p')P$.

As we have already mentioned above, in the absence of interactions $A_i^\dagger$ and A^i become the usual bosonic (commuting) or fermionic (anti-commuting) creation and annihilation operators. Then, the ZF algebra relations imply that in the free field limit the S-matrix should turn into the graded unit matrix, i.e. into the diagonal matrix with entries ± 1 depending on the statistics of the corresponding creation operator. From this point of view the relations (3.7) can be understood as a quantization (deformation) of the free oscillator algebra.

Yang–Baxter equation. In the free theory the creation operators either commute or anti-commute and, therefore, any operator monomial can be ordered in a unique way, e.g., by rearranging operators according to the momentum ordering $p_1 > p_2 > \cdots > p_n$. It is natural to require that this property of having a unique basis of the lexicographically ordered monomials holds for the interacting case as well. In the algebraic language this is known as the Poincaré–Birkhoff–Witt property. Starting from any monomial constructed from the operators $A_i^\dagger(p)$, we should be able to bring it to an ordered form in a unique way by using the defining relations (3.6) only. Consider, for instance, the product $\mathbb{A}_1^\dagger\mathbb{A}_2^\dagger\mathbb{A}_3^\dagger$, where the subscript also reflects the momentum dependence. Obviously, by using the ZF algebra relations, this monomial can be brought to the form $\mathbb{A}_3^\dagger\mathbb{A}_2^\dagger\mathbb{A}_1^\dagger$ in two different ways[21]

$$\mathbb{A}_1^\dagger\mathbb{A}_2^\dagger\mathbb{A}_3^\dagger = \mathbb{A}_3^\dagger\mathbb{A}_2^\dagger\mathbb{A}_1^\dagger S_{12}S_{13}S_{23},$$
$$\mathbb{A}_1^\dagger\mathbb{A}_2^\dagger\mathbb{A}_3^\dagger = \mathbb{A}_3^\dagger\mathbb{A}_2^\dagger\mathbb{A}_1^\dagger S_{23}S_{13}S_{12}.$$

If we require these two results to coincide without imposing new (cubic) relations between ZF operators, then the corresponding S-matrix must obey the following equation:

$$S_{23}(p_2, p_3)S_{13}(p_1, p_3)S_{12}(p_1, p_2) = S_{12}(p_1, p_2)S_{13}(p_1, p_3)S_{23}(p_2, p_3). \tag{3.9}$$

This is the Yang–Baxter equation—the fundamental equation of the factorized scattering theory.

One can show that no further constraints on the scattering matrix arise from the ordering of higher than cubic monomials provided the Yang–Baxter equation is satisfied. It is important to recognize that both the left- and right-hand side of this equation represent the three-particle scattering matrix, and the equation itself is nothing else but the factorizability condition for this S-matrix, see figure 6. Thus, the description of scattering states in terms of ZF operators with a unique basis of ordered monomials is only possible if the corresponding theory exhibits a factorizable S-matrix.

[21] Here S_{ab} denotes the standard embedding of the matrix $S(p, p')$ into the tensor product of three spaces, e.g. $S_{13}(p, p') = S_{ij}^{kl}(p, p')E_k^i \otimes \mathbb{1} \otimes E_l^j$. Note, that in general the momenta p, p' are not attached to the indices a, b.

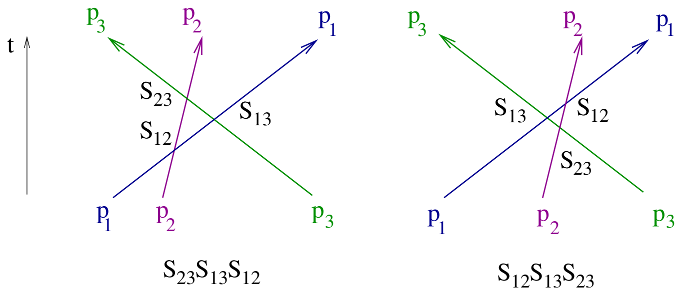

Figure 6. Factorization of the three-particle S-matrix. The result of the three-particle scattering process does not depend on the order in which two-particle scattering events take place.

Unitarity condition. In addition to the Yang–Baxter equation, consistency of the ZF algebra relations imposes further requirements on the S-matrix.

In particular, if we flip $p_1 \leftrightarrow p_2$ in the ZF algebra relation (3.6) and then pull the permutation matrix P through its left- and right-hand sides, we get

$$\begin{aligned}\mathbb{A}_2(p_2)\mathbb{A}_1(p_1) &= \mathbb{A}_1(p_1)\mathbb{A}_2(p_2)S_{21}(p_2, p_1)\\ &= \mathbb{A}_2(p_2)\mathbb{A}_1(p_1)S_{12}(p_1, p_2)S_{21}(p_2, p_1),\end{aligned}$$

where the last term was obtained by applying the ZF relation again. Thus, the S-matrix must satisfy the following property:

$$S_{12}(p_1, p_2)S_{21}(p_2, p_1) = \mathbb{1} \tag{3.10}$$

known as the unitarity condition.

Conservation laws. The fulfilment of the unitarity condition (3.10) leads to the existence in the ZF algebra of a large Abelian subalgebra. Assuming for simplicity the same dispersion relation for all the particles, this subalgebra is generated by the operators

$$\mathbb{I}_q = \int \mathrm{d}p q(p) A_i^\dagger(p) A^i(p), \tag{3.11}$$

where $q(p)$ is an arbitrary function of particle momentum. Indeed, applying the ZF algebra relations twice, we get

$$\begin{aligned}A_i^\dagger(u)A^i(u)A_j^\dagger(p) &= A_i^\dagger(u)\big[A_k^\dagger(p)A^l(u)S_{jl}^{ki}(p, u) + \delta_i^j\delta(u-p)\big]\\ &= A_n^\dagger(p)A_m^\dagger(u)A^l(u)S_{ik}^{mn}(u, p)S_{jl}^{ki}(p, u) + A_j^\dagger(p)\delta(u-p).\end{aligned}$$

In components the unitarity relation (3.10) takes the form $S_{ik}^{mn}(u, p)S_{jl}^{ki}(p, u) = \delta_j^n\delta_l^m$, and, therefore

$$\mathbb{I}_q A_i^\dagger(p) = A_i^\dagger(p)(q(p) + \mathbb{I}_q), \qquad \mathbb{I}_q A^i(p) = A^i(p)(-q(p) + \mathbb{I}_q).$$

Thus, we conclude that $\mathbb{I}_q$ for various q's do commute. Furthermore, the formulae above imply the additivity property of the commuting integrals

$$\mathbb{I}_q A_{i_1}^\dagger(p_1)\cdots A_{i_n}^\dagger(p_n)|\Omega\rangle = \left(\sum_{k=1}^n q(p_{i_k})\right) A_{i_1}^\dagger(p_1)\cdots A_{i_n}^\dagger(p_n)|\Omega\rangle.$$

In particular, as a result, we get that the Hamiltonian $\mathbb{H}$, the momentum operator $\mathbb{P}$ and the number operator $\mathbb{N}$ are given by

$$\mathbb{H} = \int \mathrm{d}p\omega(p)A_i^\dagger(p)A^i(p), \mathbb{P} = \int \mathrm{d}pp A_i^\dagger(p)A^i(p), \mathbb{N} = \int \mathrm{d}p A_i^\dagger(p)A^i(p),$$

where $\omega(p)$ is the dispersion relation which was assumed to be the same for all particles from the multiplet $\mathscr{V}$.

If particles have different dispersion relations the construction of the conservation laws admits a straightforward generalization to be discussed in due course.

Scattering and statistics. Consider an operator $(-1)^{\mathbb{N}_F}$, where we have introduced the following operator:

$$\mathbb{N}_F = \int \mathrm{d}p \epsilon_i A_i^\dagger(p) A^i(p). \tag{3.12}$$

Since $\epsilon = 0$ for bosons and $\epsilon = 1$ for fermions, $\mathbb{N}_F$ is the fermion number operator. The operator $(-1)^{\mathbb{N}_F}$ preserves the vacuum state $(-1)^{\mathbb{N}_F}|\Omega\rangle = |\Omega\rangle$ and it defines statistics of a multi-particle state

$$(-1)^{\mathbb{N}_F} \cdot A_{i_1}^\dagger(p_1)\cdots A_{i_n}^\dagger(p_n)|\Omega\rangle = (-1)^{\sum_{k=1}^n \epsilon_{i_k}} A_{i_1}^\dagger(p_1)\cdots A_{i_n}^\dagger(p_n)|\Omega\rangle.$$

Since statistics of a multi-particle state cannot change under scattering, $(-1)^{\mathbb{N}_F}$ must commute with the *S*-matrix operator $\mathbb{S}$. Pulling $(-1)^{\mathbb{N}_F}$ through the left- and the right-hand sides of the ZF relation (3.3), we get

$$(-1)^{\epsilon_i+\epsilon_j} A_i^\dagger(p_1) A_j^\dagger(p_2) = (-1)^{\epsilon_k+\epsilon_l} S_{ij}^{kl}(p_1, p_2) A_l^\dagger(p_2) A_k^\dagger(p_1).$$

The last equation leads to the following non-trivial condition for the *S*-matrix elements:

$$S_{ij}^{kl}(p_1, p_2) = (-1)^{\epsilon_i+\epsilon_j+\epsilon_k+\epsilon_l} S_{ij}^{kl}(p_1, p_2). \tag{3.13}$$

Obviously, this condition implies that for any non-vanishing $S_{ij}^{kl}(p_1, p_2)$ the sum $\epsilon_i+\epsilon_j+\epsilon_k+\epsilon_l$ is an even number: 0, 2 or 4. It is convenient to define the grading matrix Σ

$$\Sigma = (-1)^{\epsilon_i} E_i^i. \tag{3.14}$$

Then for the matrix (3.5) relation (no) can be cast in the form

$$[S(p_1, p_2), \Sigma \otimes \Sigma] = 0. \tag{3.15}$$

Thus, in the matrix language compatibility of scattering with statistics is equivalent to commutativity of $S(p_1, p_2)$ with the matrix $\Sigma \otimes \Sigma$. It is worth pointing out that the operator $\mathbb{N}_F$ does not commute with the Hamiltonian and, for this reason, the fermion number is not a conserved quantity, only $(-1)^{\mathbb{N}_F}$ is conserved.

Graded S-matrix. It is of interest to consider the following matrix:

$$S^g(p_1, p_2) = \mathbb{S}_{ij}^{kl}(p_1, p_2) E_k^i \otimes E_l^j = S_{ij}^{kl}(p_1, p_2)(-1)^{\epsilon_k\epsilon_l} E_k^i \otimes E_l^j.$$

By using the graded identity matrix (3.8), the last formula can be written as

$$S^g(p_1, p_2) = \mathbb{1}^g S(p_1, p_2), \tag{3.16}$$

where $S(p_1, p_2)$ is the matrix (3.5). The matrix S^g encodes the matrix elements of the *S*-matrix operator $\mathbb{S}$, but, contrary to *S*, it does not satisfy the Yang–Baxter equation (3.9). In what follows we will refer to S^g as the graded *S*-matrix, because it satisfies another version of (3.9) known as the graded Yang–Baxter equation.

To derive the equation, we substitute in equation (3.9) the matrix *S* expressed via S^g

$$\mathbb{1}_{23}^g S_{23}^g \mathbb{1}_{13}^g S_{13}^g \mathbb{1}_{12}^g S_{12}^g = \mathbb{1}_{12}^g S_{12}^g \mathbb{1}_{13}^g S_{13}^g \mathbb{1}_{23}^g S_{23}^g.$$

Here S^g_{ij} denotes the usual embedding of the matrix S^g into the product of three spaces. Now we note that both S and S^g obey the following identities:

$$\begin{aligned} &\mathbb{1}^g_{12}\mathbb{1}^g_{23}S_{13} = S_{13}\mathbb{1}^g_{12}\mathbb{1}^g_{23}, && \mathbb{1}^g_{12}\mathbb{1}^g_{13}S_{23} = S_{23}\mathbb{1}^g_{13}\mathbb{1}^g_{12}, \\ &\mathbb{1}^g_{12}\mathbb{1}^g_{23}S^g_{13} = S^g_{13}\mathbb{1}^g_{12}\mathbb{1}^g_{23}, && \mathbb{1}^g_{12}\mathbb{1}^g_{13}S^g_{23} = S^g_{23}\mathbb{1}^g_{13}\mathbb{1}^g_{12} \end{aligned} \tag{3.17}$$

which all follow from equation (3.13). Using these relations, equation (3.17) can be cast in the form

$$\underbrace{\mathbb{1}^g_{12}\mathbb{1}^g_{13}S^g_{23}\mathbb{1}^g_{13}\mathbb{1}^g_{12}}_{\check{S}_{23}}\underbrace{\mathbb{1}^g_{23}S^g_{13}\mathbb{1}^g_{23}}_{\check{S}_{13}}\underbrace{S^g_{12}}_{\check{S}_{12}} = \underbrace{S^g_{12}}_{\check{S}_{12}}\underbrace{\mathbb{1}^g_{23}S^g_{13}\mathbb{1}^g_{23}}_{\check{S}_{13}}\underbrace{\mathbb{1}^g_{12}\mathbb{1}^g_{13}S^g_{23}\mathbb{1}^g_{13}\mathbb{1}^g_{12}}_{\check{S}_{23}}.$$

We see that if we define the *graded embedding* of S^g into the vector product of three spaces as

$$\check{S}_{12} = S^g_{12}, \qquad \check{S}_{13} = \mathbb{1}^g_{23}S^g_{13}\mathbb{1}^g_{23}, \qquad \check{S}_{23} = \mathbb{1}^g_{12}\mathbb{1}^g_{13}S^g_{23}\mathbb{1}^g_{13}\mathbb{1}^g_{12} = S^g_{23},$$

we obtain the graded Yang–Baxter equation

$$\check{S}_{23}\check{S}_{13}\check{S}_{12} = \check{S}_{12}\check{S}_{13}\check{S}_{23}, \tag{3.18}$$

which looks the same as equation (3.9). Sometimes the matrix $\check{S}$ is referred to as the graded fermionic S-operator.

3.1.2. S-matrix and its symmetries. Now we are in position to show that the existence of a symmetry algebra of the Hamiltonian implies certain restrictions on the S-matrix.

Denote by $\mathbb{J}^{\mathbf{a}}$ the operators which generate the symmetry algebra $\mathscr{J}$

$$[\mathbb{J}^{\mathbf{a}}, \mathbb{H}] = 0, \qquad \mathbf{a} = 1, \dots, \dim\mathscr{J}.$$

In addition to $\mathbb{H}$, the symmetry generators commute with $\mathbb{P}$ and $\mathbb{N}$, and with all the higher conserved charges $\mathbb{I}_q$. The latter act diagonally in the basis of multi-particle states.

The Hilbert space created by the ZF operators carries a linear representation of $\mathscr{J}$, and since the operators $\mathbb{J}^{\mathbf{a}}$ commute with $\mathbb{N}$ and all the higher charges they must preserve the number of particles and the set of their momenta

$$\begin{aligned} &\mathbb{J}^{\mathbf{a}}\cdot|\Omega\rangle = 0, \\ &\mathbb{J}^{\mathbf{a}}\cdot A^\dagger_i(p)|\Omega\rangle = J^{\mathbf{a}}{}^j_i(p)A^\dagger_j(p)|\Omega\rangle, \\ &\mathbb{J}^{\mathbf{a}}\cdot A^\dagger_i(p_1)A^\dagger_j(p_2)|\Omega\rangle = J^{\mathbf{a}}{}^{kl}_{ij}(p_1,p_2)A^\dagger_k(p_1)A^\dagger_l(p_2)|\Omega\rangle, \\ &\dots. \end{aligned} \tag{3.19}$$

Here the tensors $J^{\mathbf{a}}{}^j_i$, $J^{\mathbf{a}}{}^{kl}_{ij}, \dots$, can be thought of as the structure constants of the symmetry algebra in one-particle, two-particle, etc representations. In general these structure constants might depend on the particle momenta. Since $\mathscr{J}$ is a superalgebra, the generator $(-1)^{\mathbb{N}_F}$ introduced in the previous section commutes with all the bosonic algebra generators and anti-commutes with the fermionic ones

$$(-1)^{\mathbb{N}_F}\cdot\mathbb{J}^{\mathbf{a}} = (-1)^{\epsilon_{\mathbf{a}}}\mathbb{J}^{\mathbf{a}}\cdot(-1)^{\mathbb{N}_F}, \tag{3.20}$$

where $\epsilon_{\mathbf{a}}$ is the degree of $\mathbb{J}^{\mathbf{a}}$. This leads to the selection rules for the corresponding structure constants

$$\begin{aligned} &J^{\mathbf{a}}{}^j_i(p) = (-1)^{\epsilon_{\mathbf{a}}+\epsilon_i+\epsilon_j}J^{\mathbf{a}}{}^j_i(p), \\ &J^{\mathbf{a}}{}^{kl}_{ij}(p_1,p_2) = (-1)^{\epsilon_{\mathbf{a}}+\epsilon_i+\epsilon_j+\epsilon_k+\epsilon_l}J^{\mathbf{a}}{}^{kl}_{ij}(p_1,p_2), \\ &\dots. \end{aligned} \tag{3.21}$$

The crucial point is that the non-Abelian symmetry algebra $\mathscr{J}$ acting on the spectrum of the Hamiltonian implies a non-trivial constraint on the scattering matrix. This constraint can be derived by acting with symmetry generators $\mathbb{J}^{\mathbf{a}}$ on the ZF algebra relations

$$\mathbb{J}^{\mathbf{a}} \cdot A_i^\dagger(p_1)A_j^\dagger(p_2)|\Omega\rangle = S_{ij}^{kl}(p_1, p_2)\mathbb{J}^{\mathbf{a}} \cdot A_l^\dagger(p_2)A_k^\dagger(p_1)|\Omega\rangle. \tag{3.22}$$

Recalling formulae (3.19), one finds that the S-matrix elements must satisfy the following invariance condition:

$$S_{kl}^{mn}(p_1, p_2)J^{\mathbf{a}}{}_{ij}^{kl}(p_1, p_2) = J^{\mathbf{a}}{}_{lk}^{nm}(p_2, p_1)S_{ij}^{kl}(p_1, p_2). \tag{3.23}$$

If we combine the symmetry generator structure constants in a matrix

$$J_{12}^{\mathbf{a}}(p_1, p_2) \equiv J^{\mathbf{a}}{}_{ij}^{kl}(p_1, p_2)E_k^i \otimes E_l^j, \tag{3.24}$$

then the invariance condition can be written as

$$S_{12}(p_1, p_2)J_{12}^{\mathbf{a}}(p_1, p_2) = J_{21}^{\mathbf{a}}(p_2, p_1)S_{12}(p_1, p_2). \tag{3.25}$$

The form of the multi-particle structure constants is determined by the symmetry algebra of a particular model. In trivial cases, $\mathscr{J}$ is a simple Lie superalgebra with (momentum-independent) structure constants in the one-particle representation

$$[\![J^{\mathbf{a}}, J^{\mathbf{b}}]\!] = t^{\mathbf{abc}}J^c,$$

where $[\![.,.]\!]$ stands for the graded commutator

$$[\![J^{\mathbf{a}}, J^{\mathbf{b}}]\!] = J^{\mathbf{a}}J^{\mathbf{b}} - (-1)^{\epsilon_{\mathbf{a}}\epsilon_{\mathbf{b}}}J^{\mathbf{b}}J^{\mathbf{a}}.$$

In this case the two-particle states can be identified with the tensor product of two one-particle states and the two-particle symmetry generators are given by[22]

$$J_{12}^{\mathbf{a}} = J^{\mathbf{a}} \otimes \mathbb{1} + \mathbb{1}^g(\mathbb{1} \otimes J^{\mathbf{a}})\mathbb{1}^g. \tag{3.26}$$

Since we work with the usual (not graded) tensor product, the second term in the right-hand side of equation (3.26) involves the graded identity which is needed for a proper account of the statistics[23]. Indeed, one can easily check that equation (3.26) defines a representation of $\mathscr{J}$ in the tensor product $\mathscr{V} \otimes \mathscr{V}$. Thus, for models with momentum-independent one-particle structure constants the invariance condition for the S-matrix reduces to the familiar matrix equations

$$\begin{aligned} (J^{\mathbf{a}} \otimes \mathbb{1} + \mathbb{1} \otimes J^{\mathbf{a}})S_{12} &= S_{12}(J^{\mathbf{a}} \otimes \mathbb{1} + \mathbb{1} \otimes J^{\mathbf{a}}) \qquad \text{for} \quad J^{\mathbf{a}} \text{ bosonic}, \\ (J^{\mathbf{a}} \otimes \mathbb{1} + \Sigma \otimes J^{\mathbf{a}})S_{12} &= S_{12}(\mathbb{1} \otimes J^{\mathbf{a}} + J^{\mathbf{a}} \otimes \Sigma) \qquad \text{for} \quad J^{\mathbf{a}} \text{ fermionic}, \end{aligned}$$

where the grading matrix Σ is defined in equation (3.14) and we specified formula (3.26) for the cases of bosonic and fermionic algebra generators. The symmetry algebra of the light-cone string sigma model is not of this simple type, however, and in our subsequent analysis we have to resort to the invariance condition (3.25).

Returning to the general situation, we assume that $\mathscr{J}$ has a non-trivial center. Then, any representation of $\mathscr{J}$ is parametrized by the particle momentum and by the corresponding values of the Lie algebra central elements (charges)[24]. Let $J^{\mathbf{a}}(p;c)$ be the generators of $\mathscr{J}$ in some representation $\mathscr{V}$, where c denotes a level set of the central elements. The generators $J^{\mathbf{a}}(p;c)$ should be thought of as matrices depending on the parameters p and c but acting in the same carrier space $\mathscr{V}$. The matrix C representing a central charge $\mathbb{C} \in \mathscr{J}$

[22] It is worth mentioning that equation (3.26) implies that the action of symmetry generators on two-particle states in equation (3.19) satisfies the Leibnitz rule. In general, however, this is not the case.

[23] In components, this formula reads as $J^{\mathbf{a}}{}_{ij}^{kl}(p_1, p_2) = J^{\mathbf{a}}{}_i^k(p_1)\delta_j^l + (-1)^{\epsilon_i\epsilon_{\mathbf{a}}}\delta_i^k J^{\mathbf{a}}{}_j^l(p_2)$.

[24] The momentum $\mathbb{P}$ commutes with all $\mathbb{J}^{\mathbf{a}}$, and therefore is central. We prefer, however, to separate $\mathbb{P}$ from other central charges due to its special role.

is $C = c\mathbb{1}$. Representations corresponding to various sets of p, c are inequivalent, because a transformation $J^{\mathbf{a}}(p; c) \to \mathfrak{g} J^{\mathbf{a}}(p; c)\mathfrak{g}^{-1}$ cannot change the value of the central charges.

Obviously, if we wish to identify $\mathscr{V}$ with a one-particle representation in the Fock space, we have to prescribe for c some fixed value (e.g., zero), as the one-particle representation is characterized by the particle momentum only and it does not involve any other continuous parameters. The structure constants in the two-particle representation can be then defined in a way similar to equation (3.26)

$$J^{\mathbf{a}}_{12}(p_1, p_2) = J^{\mathbf{a}}(p_1; c_1) \otimes \mathbb{1} + \mathbb{1}^g(\mathbb{1} \otimes J^{\mathbf{a}}(p_2; c_2))\mathbb{1}^g. \tag{3.27}$$

For a consistent interpretation of equation (3.27) as the two-particle representation, the level sets of the first and second one-particle representations should depend on the particle momenta p_1, p_2. In particular, a non-trivial situation arises when this dependence is mutually non-local—c_1 is determined by p_2 and c_2 by p_1, respectively. Plugging in equation (3.26) the matrix representatives of $\mathbb{C}$, we get

$$C_{12} = c_1\mathbb{1} \otimes \mathbb{1} + \mathbb{1}^g(\mathbb{1} \otimes c_2\mathbb{1})\mathbb{1}^g = (c_1 + c_2)\mathbb{1} \otimes \mathbb{1}. \tag{3.28}$$

Of course, this formula reflects a general fact that the value of a central charge in a tensor product representation is given by the sum of the values corresponding to the individual components of this tensor product.

As was established in subsection 2.4.2, the symmetry algebra of the light-cone sigma model has the three-dimensional center, which, in addition to $\mathbb{H}$, contains the operator $\mathbb{C}$ and its Hermitian conjugate $\mathbb{C}^\dagger$; both of them are (nonlinear) functions of the momentum operator $\mathbb{P}$. Thus, the corresponding representation theory arising in the Fock space should fit our general treatment above. Indeed, as we will show in the forthcoming sections, the simultaneous additivity of $\mathbb{C}(\mathbb{P})$ and $\mathbb{P}$ will require a realization of the two-particle representation in the form (3.27) with non-trivial functions $c_1(p_2)$ and $c_2(p_1)$.

3.1.3. General physical requirements. In a physical theory the *S*-matrix must satisfy a number of additional requirements reflecting analytic properties and discrete symmetries of the corresponding Hamiltonian. In this section we show that some of these requirements can be naturally derived by using the ZF algebra framework. We start our discussion with the condition of physical unitarity.

Physical unitarity. Since the Hamiltonian is Hermitian, the associated *S*-matrix operator $\mathbb{S}$ is unitary. To find the implications of this unitarity for the two-particle *S*-matrix $S(p_1, p_2)$, we can use the fact that the annihilation operators are Hermitian conjugate of the creation ones. Taking the Hermitian conjugation of the first line in equation (3.7), we get

$$\mathbb{A}_2(p_2)\mathbb{A}_1(p_1) = S^\dagger_{12}(p_1, p_2)\mathbb{A}_1(p_1)\mathbb{A}_2(p_2).$$

Changing $p_1 \leftrightarrow p_2$ and pulling the permutation P_{12} through the left- and the right-hand side of the last formula, we obtain

$$\mathbb{A}_1(p_1)\mathbb{A}_2(p_2) = S^\dagger_{21}(p_2, p_1)\mathbb{A}_2(p_2)\mathbb{A}_1(p_1).$$

This expression must coincide with the second line in equation (3.7) implying the relation $S^\dagger_{21}(p_2, p_1) = S_{12}(p_1, p_2)$. Using the unitarity condition (3.10), this relation can be written as

$$S^\dagger(p_1, p_2)S(p_1, p_2) = \mathbb{1} \tag{3.29}$$

meaning that $S(p_1, p_2)$ is a unitary matrix. This is the condition of physical unitarity.

Parity invariance. As was established in subsection 1.2.2 the Lagrangian of the world-sheet sigma model is invariant with respect to the parity transformation $\mathscr{P}$. This transformation acts as $\sigma \to -\sigma$ with simultaneous multiplication of fermions by i. Obviously, in the momentum space the map $\sigma \to -\sigma$ corresponds to $p \to -p$. Therefore, on one-particle states the action of $\mathscr{P}$ can be naturally defined as

$$\mathscr{P} \cdot A_i^\dagger(p)|\Omega\rangle = (-1)^{\frac{1}{2}\epsilon_i} A_i^\dagger(-p)|\Omega\rangle. \tag{3.30}$$

Here $\eta_{\mathscr{P}} = (-1)^{\frac{1}{2}\epsilon_i}$ is intrinsic parity of the particle created by $A_i^\dagger$. For a fermion $\mathscr{P}^2 = -1$ which reflects the double valuedness of the spinor representation under a rotation over an angle 2π. On multi-particle states we then have

$$\mathscr{P} \cdot |p_1, p_2, \ldots, p_n\rangle^{(in)}_{i_1,\ldots,i_n} = (-1)^{\frac{1}{2}\sum_k \epsilon_{i_k}} | - p_1, -p_2, \ldots, -p_n\rangle^{(in)}_{i_1,\ldots,i_n}.$$

Using the representation of in states in terms of the ZF operators, we can write

$$\mathscr{P} \cdot A_{i_1}^\dagger(p_1)\cdots A_{i_n}^\dagger(p_n)|\Omega\rangle = (-1)^{\frac{1}{2}\sum_k \epsilon_{i_k}} (-1)^{\sum_{k<l}\epsilon_{i_k}\epsilon_{i_l}} A_{i_n}^\dagger(-p_n)\cdots A_{i_1}^\dagger(-p_1)|\Omega\rangle,$$

where an extra statistical factor $(-1)^{\sum_{k<l}\epsilon_{i_k}\epsilon_{i_l}}$ arises due to the operator reordering. Now letting $\mathscr{P}$ act on both sides of the ZF algebra

$$\mathscr{P} \cdot A_i^\dagger(p_1)A_j^\dagger(p_2) = \mathscr{P} \cdot A_l^\dagger(p_2)A_k^\dagger(p_1)S_{ij}^{kl}(p_1, p_2), \tag{3.31}$$

and pulling $\mathscr{P}$ through, we obtain

$$\begin{aligned}(-1)^{\frac{1}{2}(\epsilon_i+\epsilon_j)+\epsilon_i\epsilon_j} A_j^\dagger(-p_2)A_i^\dagger(-p_1) &= (-1)^{\frac{1}{2}(\epsilon_k+\epsilon_l)+\epsilon_k\epsilon_l} A_k^\dagger(-p_1)A_l^\dagger(-p_2)S_{ij}^{kl}(p_1, p_2)\\ &= (-1)^{\frac{1}{2}(\epsilon_k+\epsilon_l)+\epsilon_k\epsilon_l} A_n^\dagger(-p_2)A_m^\dagger(-p_1)S_{kl}^{mn}(-p_1, -p_2)S_{ij}^{kl}(p_1, p_2),\end{aligned}$$

From here we conclude that the matrix S must obey the following condition:

$$S_{kl}^{mn}(-p_1, -p_2)S_{ij}^{kl}(p_1, p_2)(-1)^{-\epsilon_i\epsilon_j+\epsilon_k\epsilon_l+\frac{1}{2}(\epsilon_k+\epsilon_l-\epsilon_i-\epsilon_j)} = \delta_i^m\delta_j^n. \tag{3.32}$$

Since the sum $\epsilon_k + \epsilon_l + \epsilon_i + \epsilon_j$ is an even number and $\epsilon_i^2 = \epsilon_i$, we have

$$\begin{aligned}-\epsilon_i\epsilon_j + \epsilon_k\epsilon_l + \tfrac{1}{2}(\epsilon_k + \epsilon_l - \epsilon_i - \epsilon_j) &= \tfrac{1}{2}[(\epsilon_k + \epsilon_l)^2 - (\epsilon_i + \epsilon_j)^2]\\ &= \tfrac{1}{2}\underbrace{(\epsilon_k + \epsilon_l - \epsilon_i - \epsilon_j)}_{\text{even}}\underbrace{(\epsilon_k + \epsilon_l + \epsilon_i + \epsilon_j)}_{\text{even}},\end{aligned}$$

i.e. the left-hand side of the last expression is also an even number and, therefore, equation (3.32) reduces to

$$S(-p_1, -p_2) = S^{-1}(p_1, p_2). \tag{3.33}$$

This is the parity transformation rule for the S-matrix.

Time reversal. In quantum field theory the time reversal operation $\mathscr{T} : \tau \to -\tau$ is realized by means of an anti-linear, anti-unitary operator U_τ

$$U_\tau c|\Phi\rangle = \bar{c}U_\tau|\Phi\rangle, \qquad \langle\Psi|\Phi\rangle = \langle U_\tau\Phi|U_\tau\Psi\rangle.$$

To understand the implications of the symmetry under time reversal, it is convenient to start with the free field representation in terms of creation and annihilation operators as discussed in section 2.2.4. On free fields $Y^{a\dot{a}}$, $Z^{\alpha\dot{\alpha}}$, $\theta^{a\dot{\alpha}}$ and $\eta^{\alpha\dot{a}}$ the action of the anti-linear operator U_τ can be defined as follows:

$$\begin{aligned}U_\tau Y^{a\dot{a}}(\sigma, \tau)U_\tau^{-1} &= \eta_\tau Y^{a\dot{a}}(\sigma, -\tau), \qquad & U_\tau Z^{\alpha\dot{\alpha}}(\sigma, \tau)U_\tau^{-1} &= \eta_\tau Z^{\alpha\dot{\alpha}}(\sigma, -\tau),\\ U_\tau \theta^{a\dot{\alpha}}(\sigma, \tau)U_\tau^{-1} &= \eta_\tau \theta^{a\dot{\alpha}}(\sigma, -\tau), \qquad & U_\tau \eta^{\alpha\dot{a}}(\sigma, \tau)U_\tau^{-1} &= \eta_\tau \eta^{\alpha\dot{a}}(\sigma, -\tau),\end{aligned}$$

where η_τ is intrinsic time parity which depends on the type of a field. It is easy to see that for our string model it is consistent to choose $\eta_\tau = 1$ for all the fields, so that the Lagrangian density (2.61) will transform under time reversal as

$$U_\tau \mathscr{L}_2(\sigma,\tau)U_\tau^{-1} = \mathscr{L}_2(\sigma,-\tau)$$

leaving, therefore, the corresponding Lagrangian invariant. The action of time reversal on creation and annihilation operators is easy to derive by recalling the mode expansion of the corresponding fields, e.g.,

$$Y^{a\dot{a}}(\sigma,\tau) = \frac{1}{2\sqrt{2\pi}}\int \frac{\mathrm{d}p}{\sqrt{\omega_p}}\big(\mathrm{e}^{\mathrm{i}p\sigma-\mathrm{i}\omega_p\tau}a^{a\dot{a}}(p) + \mathrm{e}^{-\mathrm{i}p\sigma+\mathrm{i}\omega_p\tau}\epsilon^{ab}\epsilon^{\dot{a}\dot{b}}a^\dagger_{b\dot{b}}(p)\big)$$
$$\theta^{a\dot{\alpha}}(\sigma,\tau) = \frac{\mathrm{e}^{-\mathrm{i}\pi/4}}{\sqrt{2\pi}}\int \frac{\mathrm{d}p}{\sqrt{\omega_p}}\big(\mathrm{e}^{\mathrm{i}p\sigma-\mathrm{i}\omega_p\tau}f_p a^{a\dot{\alpha}}(p) + \mathrm{e}^{-\mathrm{i}p\sigma+\mathrm{i}\omega_p\tau}h_p\epsilon^{ab}\epsilon^{\dot{\alpha}\dot{\beta}}a^\dagger_{b\dot{\beta}}(p)\big)$$

and similarly for $Z^{\alpha\dot{\alpha}}$ and $\eta^{\alpha\dot{a}}$. Applying U_τ to these expressions, we get

$$\begin{aligned}U_\tau Y^{a\dot{a}}(\sigma,\tau)U_\tau^{-1} &= \frac{1}{2\sqrt{2\pi}}\\ &\quad\times\int \frac{\mathrm{d}p}{\sqrt{\omega_p}}\big(\mathrm{e}^{-\mathrm{i}p\sigma+\mathrm{i}\omega_p\tau}U_\tau a^{a\dot{a}}(p)U_\tau^{-1} + \mathrm{e}^{\mathrm{i}p\sigma-\mathrm{i}\omega_p\tau}\epsilon^{ab}\epsilon^{\dot{a}\dot{b}}U_\tau a^\dagger_{b\dot{b}}(p)U_\tau^{-1}\big)\\ &= Y^{a\dot{a}}(\sigma,-\tau)\end{aligned}$$

and

$$\begin{aligned}U_\tau\theta^{a\dot{\alpha}}(\sigma,\tau)U_\tau^{-1} = \frac{\mathrm{e}^{\mathrm{i}\pi/4}}{\sqrt{2\pi}}\int \frac{\mathrm{d}p}{\sqrt{\omega_p}}\big(&\mathrm{e}^{-\mathrm{i}p\sigma+\mathrm{i}\omega_p\tau}f_pU_\tau a^{a\dot{\alpha}}(p)U_\tau^{-1}\\ &+\mathrm{e}^{\mathrm{i}p\sigma-\mathrm{i}\omega_p\tau}h_p\epsilon^{ab}\epsilon^{\dot{\alpha}\dot{\beta}}U_\tau a^\dagger_{b\dot{\beta}}(p)U_\tau^{-1}\big) = \theta^{a\dot{\alpha}}(\sigma,-\tau).\end{aligned}$$

From here we deduce the transformation law for creation and annihilation operators

$$\begin{aligned}&U_\tau a^{a\dot{a}}(p)U_\tau^{-1} = a^{a\dot{a}}(-p), && U_\tau a^\dagger_{b\dot{b}}(p)U_\tau^{-1} = a^\dagger_{b\dot{b}}(-p),\\ &U_\tau a^{a\dot{\alpha}}(p)U_\tau^{-1} = -\mathrm{i}a^{a\dot{\alpha}}(-p), && U_\tau a^\dagger_{b\dot{\beta}}(p)U_\tau^{-1} = \mathrm{i}a^\dagger_{b\dot{\beta}}(-p).\end{aligned}\tag{3.34}$$

It is interesting to note that in classical theory and before gauge fixing, time reversal can be defined in a way similar to parity reversal, namely, $\tau \to -\tau$ with simultaneous multiplication of fermions θ and η by i and − i, respectively. We see that in the gauge-fixed quantum theory, with the well-defined Hamiltonian and the canonical structure, these are fermionic creation and annihilation operators that under time reversal are multiplied by i or −i, rather then θ and η.

Formulae (3.34) derived for free theory suggest how to define the time reversal operation $\mathscr{T}$ in interacting theory. On a one-particle state created by a ZF operator we define an action of $\mathscr{T}$ as

$$\mathscr{T}\cdot A_i^\dagger(p)|\Omega\rangle = i^{\epsilon_i}A_i^\dagger(-p)|\Omega\rangle. \tag{3.35}$$

Since $\mathscr{T}$ maps $\tau \to -\tau$, it interchanges asymptotic past and future and, for this reason, its action on multi-particle states is given by

$$\mathscr{T}\cdot|p_1,p_2,\ldots,p_n\rangle^{(in)}_{i_1,\ldots,i_n} = (-1)^{\frac{1}{2}\sum_k\epsilon_{i_k}}|-p_1,-p_2,\ldots,-p_n\rangle^{(out)}_{i_1,\ldots,i_n}.$$

Representing *in* and *out* states in terms of the ZF operators, the last formula can be written as

$$\mathscr{T}\cdot A^\dagger_{i_1}(p_1)\cdots A^\dagger_{i_n}(p_n)|\Omega\rangle = (-1)^{\frac{1}{2}\sum_k\epsilon_{i_k}}A^\dagger_{i_1}(-p_1)\cdots A^\dagger_{i_n}(-p_n)|\Omega\rangle.$$

Commuting $\mathscr{T}$ through both sides of the ZF algebra relations (3.3), one gets

$$(-1)^{\frac{1}{2}(\epsilon_i+\epsilon_j)}A_i^\dagger(-p_1)A_j^\dagger(-p_2) = (-1)^{\frac{1}{2}(\epsilon_k+\epsilon_l)}A_l^\dagger(-p_2)A_k^\dagger(-p_1)S_{ij}^{*kl}(p_1,p_2),$$

where S_{ij}^{*kl} stands for the complex conjugate of the S-matrix element S_{ij}^{kl}, and we have taken into account that $\mathscr{T}$ is an anti-unitary operator. Permuting the ZF operators in the right-hand side of the last relation, one obtains

$$(-1)^{\frac{1}{2}(\epsilon_i+\epsilon_j)}A_i^\dagger(-p_1)A_j^\dagger(-p_2) = (-1)^{\frac{1}{2}(\epsilon_k+\epsilon_l)}A_n^\dagger(-p_1)A_m^\dagger(-p_2)S_{lk}^{mn}(-p_1,-p_2)S_{ij}^{*kl}(p_1,p_2).$$

Thus, invariance of the theory under time reversal leads to the following equation for the matrix elements of the S-matrix:

$$S_{ij}^{*kl}(p_1,p_2)S_{lk}^{mn}(-p_1,-p_2)(-1)^{\frac{1}{2}(\epsilon_k+\epsilon_l-\epsilon_i-\epsilon_j)} = \delta_i^n\delta_j^m. \tag{3.36}$$

According to our discussion of the parity transform,

$$(-1)^{\epsilon_i\epsilon_j+\epsilon_k\epsilon_l} = (-1)^{\frac{1}{2}(\epsilon_k+\epsilon_l-\epsilon_i-\epsilon_j)}.$$

Therefore, in the matrix form equation (3.36) reads as

$$\mathbb{1}^g S_{12}^*(p_1,p_2)\mathbb{1}^g S_{21}(-p_2,-p_1) = \mathbb{1}. \tag{3.37}$$

This is the condition on the two-particle S-matrix implied by the time reversal invariance.

Unitarity condition (3.10) in conjunction with parity invariance (3.33) and physical unitarity (3.29) allows one to rewrite the last formula in the following form:

$$S^t(p_1,p_2) = \mathbb{1}^g S(p_1,p_2)\mathbb{1}^g, \tag{3.38}$$

that can be viewed as the consequence of the combined parity and time reversal invariance.

Charge conjugation. As before, we assume that particles (one-particle asymptotic states) transform in some representation $\mathscr{V}$ of the symmetry algebra $\mathscr{J}$. Let $\mathscr{B}$ be the bosonic subalgebra of $\mathscr{J}$. If a theory is invariant under charge conjugation then there are two possibilities—either a representation of $\mathscr{B}$ in $\mathscr{V}$ is reducible and consists of two representations conjugate to each other or it is self-conjugate.

In the first case we have $\mathscr{V} = \mathscr{W}\oplus\mathscr{W}^*$, where the first and the second components correspond to particles and anti-particles, respectively[25]. If $\mathscr{D}$ is a matrix realization of the group corresponding to $\mathscr{B}$ which acts in the space $\mathscr{W}$, then anti-particles transform in the conjugate representations $\mathscr{W}^*$ with the matrix realization $\mathscr{D}^*$. Note that for unitary groups the conjugate representation coincides with the contragradient representation: $(\mathscr{D}^t)^{-1} = \mathscr{D}^*$. Charge conjugation is understood as a transfer

$$C : \mathscr{W} \to \mathscr{W}^*.$$

In general, C belongs to the group of outer automorphisms of $\mathscr{B}$.

In the second case, the representation $\mathscr{V}$ is self-conjugate which means that $\mathscr{V}^*$ is equivalent to $\mathscr{V}$. For instance, if the bosonic subalgebra of $\mathscr{J}$ is $\mathfrak{su}(2)$, then

$$\mathscr{D}^* = C\mathscr{D}C^{-1},$$

where, according to equation (1.134), $C = \epsilon$ is an internal automorphism. Obviously, under these circumstances, invariance under charge conjugation does not lead to any new restrictions on the form of the two-particle S-matrix beyond those implied by $\mathscr{J}$. This is precisely the situation we encounter for the string sigma model.

[25] The reader might have in mind, for instance, quarks and anti-quarks which transform in fundamental and anti-fundamental irreps of SU(3).

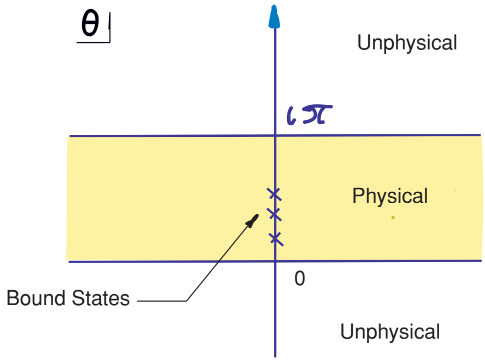

Figure 7. Physical strip on the rapidity plane of a two-dimensional relativistic field theory.

Crossing symmetry. So far we were considering the obvious kinematical symmetries of the Hamiltonian. Now we introduce a new type of dynamical symmetry which manifests itself in the scattering process as a possibility of replacing a particle with its anti-particle. In relativistic theories this kind of symmetry is known as crossing.

Recall that in two-dimensional Lorentz-invariant models the particle momentum p and the energy H can be parametrized by a single rapidity variable θ

$$p = \sinh\theta, \qquad H = \cosh\theta, \tag{3.39}$$

which provides a solution to the relativistic dispersion relation

$$H^2 - p^2 = 1, \tag{3.40}$$

where for simplicity we assumed a particle of unit mass. Invariance under Lorentz transformations requires the two-particle S-matrix to depend on the difference of the particle rapidities: $S(p_1, p_2) = S(\theta_1 - \theta_2)$.

To describe all states in a theory, including bound states, the rapidity variable should be continued to the complex plane. The already mentioned crossing symmetry transformation corresponds to the shift $\theta \to \theta + \mathrm{i}\pi$, because the momentum and energy change a sign

$$\theta \to \theta + \mathrm{i}\pi : \qquad p \to -p, \qquad H \to -H,$$

which signifies a transition to the corresponding anti-particle. The difference $\theta = \theta_1 - \theta_2$ takes value in the strip $0 \leqslant \mathrm{Im}\,\theta < \pi$ and $-\infty < \mathrm{Re}\,\theta < \infty$, which is called the physical strip of a relativistic field theory, see figure 7.

Crossing symmetry leads to further constraints on the scattering matrix. Although the string sigma model does not have Lorentz invariance on the world-sheet, as we will show, the corresponding scattering theory is compatible with the assumption of crossing.

We will reserve a detailed discussion of crossing symmetry for subsection 3.4.2. Here our goal will be to demonstrate that the crossing symmetry requirement for the S-matrix naturally follows from an additional invariance condition of the ZF algebra.

This invariance condition is related to the possibility of exchanging in the ZF relations creation and annihilation operators corresponding to one of the two particles. More precisely, we define the following transformation:

$$\mathbb{A}^\dagger(p) \to \mathbb{B}^\dagger(p) = \mathbb{A}^t(-p)\mathscr{C}, \qquad \mathbb{A}(p) \to \mathbb{B}(p) = \mathscr{C}^\dagger \mathbb{A}^{\dagger t}(-p), \tag{3.41}$$

where $\mathscr{C}$ is a constant matrix and superscript t means transposition. We require that under this map the ZF algebra relations (3.7) for $p_1 \neq p_2$ transform into themselves. More precisely, if we first replace in the algebra relations $\mathbb{A}$ by $\mathbb{B}$ for one of the particles, and further use formulae (3.41) to express $\mathbb{B}$ via $\mathbb{A}$, we should recover for $\mathbb{A}$ the same relations. Under the assumption $p_1 > p_2$ the delta-function does not contribute which makes it possible to map by means of equation (3.41) the exchange relations of $\mathbb{A}(p_1)$ and $\mathbb{A}(p_2)$ to that of $\mathbb{A}(p_1)$ and $\mathbb{A}^\dagger(p_2)$.

Note that flipping the sign of p under (3.41) is dictated by the compatibility of equation (3.41) with the algebra relations

$$\mathbb{P}A^\dagger = A^\dagger(\mathbb{P}+p), \qquad \mathbb{P}A = A(\mathbb{P}-p),$$

i.e. the operators $\mathbb{A}^\dagger$ and $\mathbb{B}^\dagger$ are required to commute with $\mathbb{P}$ in the same way.

Application of the crossing symmetry transformation to the *S*-matrix requires a certain care. Crossing symmetry does not only change the sign of p but it also changes a branch of the dispersion relation sending H to $-H$. To correctly implement the action of crossing symmetry, the *S*-matrix could be treated as a function of both the particle momenta and the particle energies: $S(p_1, H_1; p_2, H_2)$. Of course, on any given branch the *S*-matrix becomes a function of particle momenta only. Even then the *S*-matrix is not a meromorphic function of p_i and H_i, and one still should specify additional cuts in the pH-planes and choose a proper branch. Crossing, e.g. the first particle then invokes the following transformation:

$$S(p_1, H_1; p_2, H_2) \to S^{c_1}(-p_1, -H_1; p_2, H_2),$$

and applying it twice one does not end up with the original *S*-matrix: $(S^{c_1})^{c_1} \neq S$.

Although below we will not specify explicitly the branch dependence of the *S*-matrix, it is precisely in this sense we understand the action of crossing on *S*. Clearly, finding an analog of the rapidity variable which uniformizes a given dispersion relation and makes the *S*-matrix a meromorphic function would greatly simplify the treatment of crossing symmetry as it resolves the ambiguities of *S* related to the choice of a branch. For the string sigma model at hand, such a uniformization rendering the crossing symmetric world-sheet *S*-matrix a meromorphic function is unknown. We will return to this important issue in subsection 3.2.4.

Meanwhile, we find that invariance of the ZF algebra under map (3.41) implies the following equations:

$$\begin{aligned}
&\mathscr{C}_1^{-1} S_{12}^{t_1}(p_1, p_2)\mathscr{C}_1 S_{12}(-p_1, p_2) = \mathbb{1},\\
&\mathscr{C}_2^{-1} S_{21}^{t_2}(p_2, p_1)\mathscr{C}_2 S_{21}(-p_2, p_1) = \mathbb{1}.
\end{aligned} \tag{3.42}$$

Here t_1 and t_2 mean the transposition in the first and second space, respectively, $\mathscr{C}_1 = \mathscr{C} \otimes \mathbb{1}, \mathscr{C}_2 = \mathbb{1} \otimes \mathscr{C}$. In fact, these two equations are equivalent: the first turns into the second after applying the permutation and exchanging p_1 and p_2. Provided $\mathscr{C}$ is known, equations (3.42) represent a further non-trivial constraint on the *S*-matrix.

There is an alternative way to obtain equations (3.42). Without loss of generality we assume that $\mathscr{C}^\dagger\mathscr{C} = \mathbb{1}$ and consider the following singlet (row × matrix × column):

$$\mathrm{I}(p) = \mathbb{A}^\dagger(-p)\mathscr{C}^{-1}\mathbb{A}^{\dagger t}(p).$$

This operator commutes with $\mathbb{P}$ and, when applied to the vacuum, produces a state with zero momentum. We require this operator to have trivial scattering with all operators $A^\dagger$

$$\mathrm{I}_1(p_1)\mathbb{A}_2^\dagger(p_2) = \mathbb{A}_2^\dagger(p_2)\,\mathrm{I}_1(p_1).$$

This gives

$$
\begin{aligned}
\underbrace{\mathbb{A}_1^\dagger(-p_1)\mathscr{C}_1^{-1}\mathbb{A}_1^{\dagger t}(p_1)}_{\mathrm{I}_1(p_1)}\mathbb{A}_2^\dagger(p_2) &= \mathbb{A}_1^\dagger(-p_1)\mathscr{C}_1^{-1}\left[\mathbb{A}_1^\dagger(p_1)\mathbb{A}_2^\dagger(p_2)\right]^{t_1}\\
&= \mathbb{A}_1^\dagger(-p_1)\mathscr{C}_1^{-1}\left[\mathbb{A}_2^\dagger(p_2)\mathbb{A}_1^\dagger(p_1)S_{12}(p_1,p_2)\right]^{t_1}\\
&= \mathbb{A}_1^\dagger(-p_1)\mathbb{A}_2^\dagger(p_2)\mathscr{C}_1^{-1}S_{12}^{t_1}(p_1,p_2)\mathbb{A}_1^{\dagger t_1}(p_1)\\
&= \mathbb{A}_2^\dagger(p_2)\underbrace{\mathbb{A}_1^\dagger(-p_1)S_{12}(-p_1,p_2)\mathscr{C}_1^{-1}S_{12}^{t_1}(p_1,p_2)\mathbb{A}_1^{\dagger t_1}(p_1)}_{\mathrm{I}_1(p_1)},
\end{aligned}
$$

i.e. we must require

$$
S_{12}(-p_1,p_2)\mathscr{C}_1^{-1}S_{12}^{t_1}(p_1,p_2) = \mathscr{C}_1^{-1},
$$

which is equivalent to equations (3.42). The concrete form of the matrix $\mathscr{C}$ will be found in subsection 3.4.2.

Summary. We conclude this section by summarizing the basic physical requirements for the scattering matrix:

- *Generalized physical unitarity*
$$S(p_1^*,p_2^*)^\dagger\cdot S(p_1,p_2) = \mathbb{1}$$
- *Parity invariance*
$$S(-p_1,-p_2) = S^{-1}(p_1,p_2).$$
- *Time reversal invariance*
$$S(p_1,p_2)^t = \mathbb{1}^g S(p_1,p_2)\mathbb{1}^g.$$
- *Crossing symmetry*
$$S^{c_1}(p_1,p_2)S(-p_1,p_2) = \mathbb{1}, \qquad S^{c_2}(p_1,p_2)S(p_1,-p_2) = \mathbb{1}.$$

Some comments are in order. For real values of momenta the S-matrix must be unitary. If the momenta are complex, usual unitarity is replaced by the generalized unitarity condition above, where p^* stands for the complex conjugate momentum. The time reversal invariance condition presented here assumes parity invariance and physical unitarity. Finally, the crossing symmetry relates the anti-particle-to-particle scattering matrix S^{c_1} to that of particle-to-particle and it holds for the properly normalized S only.

3.2. Fundamental representation of $\mathfrak{su}(2|2)_c$

In this section we will describe the fundamental representation of the centrally extended superalgebra $\mathfrak{su}(2|2)_c$. For the reader's convenience, we repeat the Lie algebra defining relations (see, section 2.4.2 for notations)

$$
\begin{aligned}
&\left[\mathbb{L}_a{}^b,\mathbb{J}_c\right] = \delta_c^b\mathbb{J}_a - \tfrac{1}{2}\delta_a^b\mathbb{J}_c, \qquad \left[\mathbb{R}_\alpha{}^\beta,\mathbb{J}_\gamma\right] = \delta_\gamma^\beta\mathbb{J}_\alpha - \tfrac{1}{2}\delta_\alpha^\beta\mathbb{J}_\gamma,\\
&\left[\mathbb{L}_a{}^b,\mathbb{J}^c\right] = -\delta_a^c\mathbb{J}^b + \tfrac{1}{2}\delta_a^b\mathbb{J}^c, \qquad \left[\mathbb{R}_\alpha{}^\beta,\mathbb{J}^\gamma\right] = -\delta_\alpha^\gamma\mathbb{J}^\beta + \tfrac{1}{2}\delta_\alpha^\beta\mathbb{J}^\gamma,\\
&\left\{\mathbb{Q}_\alpha{}^a,\mathbb{Q}_b^{\dagger\beta}\right\} = \delta_b^a\mathbb{R}_\alpha{}^\beta + \delta_\alpha^\beta\mathbb{L}_b{}^a + \tfrac{1}{2}\delta_b^a\delta_\alpha^\beta\mathbb{H},\\
&\left\{\mathbb{Q}_\alpha{}^a,\mathbb{Q}_\beta{}^b\right\} = \epsilon_{\alpha\beta}\epsilon^{ab}\mathbb{C}, \qquad \left\{\mathbb{Q}_a^{\dagger\alpha},\mathbb{Q}_b^{\dagger\beta}\right\} = \epsilon_{ab}\epsilon^{\alpha\beta}\mathbb{C}^\dagger.
\end{aligned}
\tag{3.43}
$$

In section 2.4.2 these relations have been derived by studying the Poisson bracket of the Noether charges of the string sigma-model in the light-cone gauge. It was found there that upon going off-shell the algebra $\mathfrak{su}(2|2)$ receives the central extension by two central charges $\mathbb{C}$ and $\mathbb{C}^\dagger$. To make our treatment more general, we will for a moment assume that $\mathbb{C}$ and $\mathbb{C}^\dagger$ are independent.

3.2.1. Matrix realization. Introduce a basis of the four-dimensional fundamental representation

$$|e_M\rangle = \begin{cases} |e_a\rangle \\ |e_\alpha\rangle. \end{cases}$$

Here $\epsilon_a = 0$ for $a = 1, 2$ and $\epsilon_\alpha = 1$ for $\alpha = 3, 4$. On these basis vectors the rotation generators of (3.43) are realized as

$$\begin{aligned} &\mathbb{L}_a{}^b|e_c\rangle = \delta_c^b|e_a\rangle - \tfrac{1}{2}\delta_a^b|e_c\rangle \qquad \mathbb{R}_\alpha{}^\beta|e_a\rangle = 0 \\ &\mathbb{L}_a{}^b|e_\alpha\rangle = 0 \qquad \mathbb{R}_\alpha{}^\beta|e_\gamma\rangle = \delta_\gamma^\beta|e_\alpha\rangle - \tfrac{1}{2}\delta_\alpha^\beta|e_\gamma\rangle. \end{aligned} \tag{3.44}$$

The supersymmetry generators will then be represented as

$$\begin{aligned} &\mathbb{Q}_\alpha{}^a|e_b\rangle = a\delta_a^b|e_\alpha\rangle \qquad \mathbb{Q}^{\dagger}{}_a^{\alpha}|e_b\rangle = c\epsilon_{ab}\epsilon^{\alpha\beta}|e_\beta\rangle \\ &\mathbb{Q}_\alpha{}^a|e_\beta\rangle = b\epsilon_{\alpha\beta}\epsilon^{ab}|e_b\rangle \qquad \mathbb{Q}^{\dagger}{}_a^{\alpha}|e_\beta\rangle = d\delta_\beta^\alpha|e_a\rangle. \end{aligned} \tag{3.45}$$

Here a, b, c, d are complex numbers parametrizing a fundamental irrep. One can check that the algebra relations (3.43) are satisfied provided these numbers satisfy the following relation:

$$ad - bc = 1. \tag{3.46}$$

The values of the central elements are found to be

$$\mathbb{H}|e_M\rangle = (ad + bc)|e_M\rangle, \qquad \mathbb{C}|e_M\rangle = ab|e_M\rangle, \qquad \mathbb{C}^\dagger|e_M\rangle = cd|e_M\rangle. \tag{3.47}$$

In addition, if we require this representation to be unitary, then the parameters have to satisfy the conditions

$$d^* = a, \qquad c^* = b.$$

In unitary representations, $\mathbb{H}$ is Hermitian and $\mathbb{C}$ is the Hermitian conjugate of $\mathbb{C}^\dagger$.

It is convenient to combine the parameters describing the set of fundamental unitary representations into the following matrix:

$$h = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Since this matrix obeys the relation $h^\dagger \rho h = \rho$, where $\rho = \mathrm{diag}(1, -1)$ and it has unit determinant, it can be thought of as an element of the three-dimensional $\mathrm{SU}(1, 1)$ group. Not all the values of the central charges are allowed, however. Indeed, equations (3.46) and (3.47) imply that

$$H^2 - 4C\bar{C} = 1. \tag{3.48}$$

This is the so-called shortening condition which defines an atypical (short) multiplet of $\mathfrak{su}(2|2)_C$ of dimension four. Thus, the space of central charges corresponding to atypical four-dimensional multiplets is parametrized by one real variable, which is H, and by the phase of C.

Any element of $\mathrm{SU}(1, 1)$ gives rise to the central charges H and C obeying equation (3.48). On the other hand, given the charges (3.47) satisfying equation (3.48), the representation parameters are not specified uniquely because a $\mathrm{U}(1)$ automorphism

$$h \to \begin{pmatrix} a\,\mathrm{e}^{\mathrm{i}\varphi} & b\,\mathrm{e}^{-\mathrm{i}\varphi} \\ c\,\mathrm{e}^{\mathrm{i}\varphi} & d\,\mathrm{e}^{-\mathrm{i}\varphi} \end{pmatrix}$$

does not change the value of the charges and merely reflects a choice of basis. Factoring out this $\mathrm{U}(1)$ subgroup, we obtain a two-sheeted hyperboloid $\mathrm{SU}(1, 1)/\mathrm{U}(1)$ which is described by

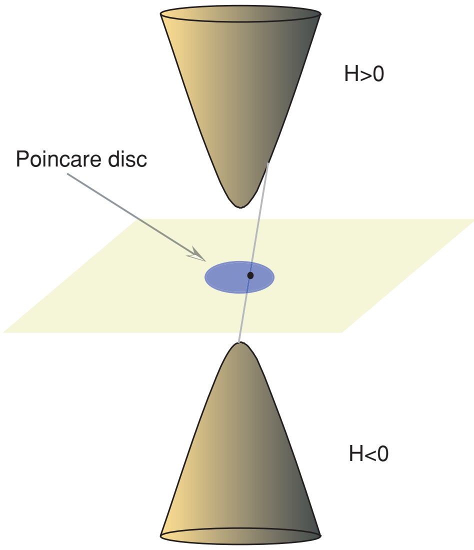

Figure 8. Two branches of the dispersion relation corresponding to $H > 0$ and $H < 0$, respectively. The Poincaré (blue) disk represents the stereographic projection of an upper sheet on the complex plane through the origin.

equation (3.48). The upper sheet $H > 0$ corresponds to positive energy unitary representations, while the lower sheet corresponds to anti-unitary representations, see figure 8.

Finally, for the reader's convenience, we describe the representation (3.44), (3.45) in terms of 4×4 matrix unities

$$\begin{aligned} &{L_1}^2 = E_1^2, && {R_3}^4 = E_3^4, \\ &{L_2}^1 = E_2^1, && {R_4}^3 = E_4^3, \\ &{L_1}^1 = \tfrac{1}{2}\left(E_1^1 - E_2^2\right) = -{L_2}^2, && {R_3}^3 = \tfrac{1}{2}\left(E_3^3 - E_4^4\right) = -{R_4}^4 \end{aligned} \tag{3.49}$$

and

$$\begin{aligned} &Q_3^1 = aE_3^1 + bE_2^4, && Q_1^{\dagger 3} = cE_4^2 + dE_1^3, \\ &Q_4^1 = aE_4^1 - bE_2^3, && Q_1^{\dagger 4} = -cE_3^2 + dE_1^4, \\ &Q_3^2 = aE_3^2 - bE_1^4, && Q_2^{\dagger 3} = -cE_4^1 + dE_2^3, \\ &Q_4^2 = aE_4^2 + bE_1^3, && Q_2^{\dagger 4} = cE_3^1 + dE_2^4. \end{aligned} \tag{3.50}$$

3.2.2. Outer automorphisms. Over the complex field, the superalgebra $\mathfrak{su}(2|2)_c$ admits a group of outer automorphisms isomorphic to SL(2). This group acts on the supercharges in the following way:

$$\begin{aligned} &\widetilde{\mathbb{Q}}_\alpha{}^a = u_1 \mathbb{Q}_\alpha{}^a - u_2 \epsilon^{ac} \mathbb{Q}_c^{\dagger\,\gamma} \epsilon_{\gamma\alpha}, && \widetilde{\mathbb{Q}}_a^{\dagger\,\alpha} = v_1 \mathbb{Q}_a^{\dagger\,\alpha} - v_2 \epsilon^{\alpha\beta} \mathbb{Q}_\beta{}^b \epsilon_{ba}, \\ &\mathbb{Q}_\alpha{}^a = v_1 \widetilde{\mathbb{Q}}_\alpha{}^a + u_2 \epsilon^{ac} \widetilde{\mathbb{Q}}_c^{\dagger\,\gamma} \epsilon_{\gamma\alpha}, && \mathbb{Q}_a^{\dagger\,\alpha} = u_1 \widetilde{\mathbb{Q}}_a^{\dagger\,\alpha} + v_2 \epsilon^{\alpha\beta} \widetilde{\mathbb{Q}}_\beta{}^b \epsilon_{ba}, \end{aligned} \tag{3.51}$$

where the coefficients u_i, v_i satisfy the condition

$$u_1 v_1 - u_2 v_2 = 1, \tag{3.52}$$

which guarantees that $\widetilde{\mathbb{Q}}$ and $\widetilde{\mathbb{Q}}^\dagger$ obey the same algebra relations (3.43) but with the new central elements given by

$$\begin{aligned}
\widetilde{\mathbb{H}} &= (u_1 v_1 + u_2 v_2)\mathbb{H} + 2u_1 v_2 \mathbb{C} + 2u_2 v_1 \mathbb{C}^\dagger, \\
\widetilde{\mathbb{C}} &= u_1^2 \mathbb{C} + u_2^2 \mathbb{C}^\dagger + u_1 u_2 \mathbb{H}, \\
\widetilde{\mathbb{C}}^\dagger &= v_1^2 \mathbb{C}^\dagger + v_2^2 \mathbb{C} + v_1 v_2 \mathbb{H}.
\end{aligned} \tag{3.53}$$

The transformation parameters u_i, v_i are combined into a complex 2×2-matrix

$$\begin{pmatrix} u_1 & u_2 \\ v_2 & v_1 \end{pmatrix},$$

which, due to equation (3.52), has unit determinant. This establishes an isomorphism of the outer automorphism group to SL(2). Restriction to the unitary representations of the real form $\mathfrak{su}(2|2)_c$ will require one to replace SL(2) with its real form SU(1, 1); the latter is defined by imposing the following two conditions:

$$v_1^* = u_1, \qquad v_2^* = u_2.$$

Further, one can see that the action (3.53) leaves the following combination of charges invariant:

$$\mathbb{H}^2 - 4\mathbb{C}\mathbb{C}^\dagger \equiv \mathscr{R}^2.$$

The invariant $\mathscr{R}^2$ classifies the orbits of SU(1, 1) in the space of central charges. They can be of three types depending on the value of $\mathscr{R}^2$—a two-sheeted hyperboloid for $\mathscr{R}^2 > 0$, a one-sheeted hyperboloid $\mathscr{R}^2 < 0$ and a cone for $\mathscr{R}^2 = 0$. We are interested in the $\mathscr{R}^2 > 0$ orbits only, because these orbits correspond to the positive and negative energy unitary representations of $\mathfrak{su}(2|2)_c$.

The outer automorphism group allows one to establish a connection between the positive/negative energy (highest/lowest weight) representations of $\mathfrak{su}(2|2)_c$ and those of the usual (non-extended) algebra $\mathfrak{su}(2|2)$. Indeed, starting with an irrep of $\mathfrak{su}(2|2)_c$ characterized by some values of $\mathbb{C}, \mathbb{C}^\dagger, \mathbb{H}$ with $\mathscr{R}^2 > 0$ and choosing the parameters u_i, v_i appropriately, one can always make the charges $\widetilde{\mathbb{C}}$ and $\widetilde{\mathbb{C}}^\dagger$ vanishing. Thus, the transformed representation is the one for usual $\mathfrak{su}(2|2)$ with $\widetilde{\mathbb{H}}$ equal to

$$\widetilde{\mathbb{H}} = \pm\sqrt{\mathbb{H}^2 - 4\mathbb{C}\mathbb{C}^\dagger},$$

where the sign in front of the square root correlates with the sign of $\mathbb{H}$. The inverse statement is also true: any irreducible representation of the centrally extended algebra with $\mathscr{R}^2 > 0$ can be obtained from a representation of the usual $\mathfrak{su}(2|2)$ algebra with $\mathbb{C} = \mathbb{C}^\dagger = 0$.

Let us now describe in more detail the action of the outer automorphism group on the fundamental irreps of $\mathfrak{su}(2|2)_c$. Under this action the matrix h encoding the representation parameters undergo the right shift by an SU(1, 1)-matrix

$$h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} u_1 & u_2 \\ v_2 & v_1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{3.54}$$

According to the discussion above, SU(1, 1) acts transitively on each sheet of the two-sheeted hyperboloid $\mathscr{R}^2 = 1$. The tip of the upper sheet corresponds to the special irrep with vanishing values for $\mathbb{C}$ and $\mathbb{C}^\dagger$, and $\mathbb{H}$ equal to unity. This representation is nothing else but the unique fundamental positive energy representation of the non-extended algebra $\mathfrak{su}(2|2)$.

3.2.3. Parameterizations of a, b, c, d. The parameters a, b, c, d depend on the string tension g and the world-sheet momenta p. To find their explicit dependence, we take into account that the central charges are expressed via the momentum $\mathbb{P}$ by equation (2.128). Therefore, the parameters satisfy the following relations:

$$ab = \frac{\mathrm{i}g}{2}(\mathrm{e}^{\mathrm{i}p} - 1)\,\mathrm{e}^{2\mathrm{i}\xi}, \qquad cd = \frac{g}{2\mathrm{i}}(\mathrm{e}^{-\mathrm{i}p} - 1)\,\mathrm{e}^{-2\mathrm{i}\xi}, \qquad H = ad + bc = 2ad - 1, \tag{3.55}$$

where p is the value of the world-sheet momentum $\mathbb{P}$ on the representation.

The shortening condition (3.48) implies that the energy depends on p only and it leads to the following dispersion relation for particles from the fundamental $\mathfrak{su}(2|2)_C$ multiplet:

$$H^2 = 1 + 4g^2 \sin^2 \frac{p}{2}. \tag{3.56}$$

To simplify our further treatment, we assume that the representation is unitary. In this case the parameters g, p, ξ are real, H is positive, and the equations (3.55), (3.46) allow one to parametrize a, b, c, d as

$$\begin{aligned} a &= \eta\,\mathrm{e}^{\mathrm{i}\xi}\,\mathrm{e}^{\mathrm{i}\varphi}, & b &= \frac{g}{2}(\mathrm{e}^{\mathrm{i}p} - 1)\frac{\mathrm{i}\,\mathrm{e}^{\mathrm{i}\xi}}{\eta}\,\mathrm{e}^{-\mathrm{i}\varphi}, \\ d &= \eta\mathrm{e}^{-\frac{\mathrm{i}p}{2}}\,\mathrm{e}^{-\mathrm{i}\xi}\,\mathrm{e}^{-\mathrm{i}\varphi}, & c &= \frac{g}{2}(\mathrm{e}^{-\mathrm{i}p} - 1)\,\mathrm{e}^{\frac{\mathrm{i}p}{2}}\frac{\mathrm{e}^{-\mathrm{i}\xi}}{\mathrm{i}\eta}\,\mathrm{e}^{\mathrm{i}\varphi}, \end{aligned} \tag{3.57}$$

where for unitary representations φ is an arbitrary real number, and η is expressed through the momentum p and the energy $H = \sqrt{1 + 4g^2 \sin^2 \frac{p}{2}}$ as follows:

$$\eta = \mathrm{e}^{\frac{\mathrm{i}p}{4}}\sqrt{\frac{H+1}{2}}. \tag{3.58}$$

In the last formula the prefactor $\mathrm{e}^{\frac{\mathrm{i}p}{4}}$ may look rather artificial. Nevertheless, it plays an important role in what follows, in particular, its presence will make η a meromorphic function on the rapidity torus we introduce in the following subsection.

The fundamental representation is completely determined by the parameters g, p, ξ. The parameter φ just reflects a freedom in the choice of the basis vectors $|e_M\rangle$, and in what follows we set it to zero by proper rescaling of $|e_M\rangle$. Then, formulae (3.57) render the parameters a, b, c, d of the fundamental representation as functions of the three independent parameters g, p, ξ.

Another convenient parametrization is obtained by replacing the momentum p with two new parameters x^+, x^-. They are related to p as

$$\frac{x^+}{x^-} = \mathrm{e}^{\mathrm{i}p}, \tag{3.59}$$

and satisfy the constraint

$$x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{2\mathrm{i}}{g}. \tag{3.60}$$

One can show that a, b, c, d are then expressed through g, $x^\pm$ and ξ in the following way (we set $\varphi = 0$):

$$a = \eta\mathrm{e}^{\mathrm{i}\xi}, \qquad b = -\eta\,\frac{\mathrm{e}^{-\frac{p}{2}}}{x^-}\,\mathrm{e}^{\mathrm{i}\xi}, \qquad c = -\eta\,\frac{\mathrm{e}^{-\mathrm{i}\xi}}{x^+}, \qquad d = \eta\,\mathrm{e}^{-\frac{\mathrm{i}p}{2}}\,\mathrm{e}^{-\mathrm{i}\xi}, \tag{3.61}$$

where the parameter η is given by

$$\eta = \mathrm{e}^{\frac{\mathrm{i}p}{4}}\sqrt{\frac{\mathrm{i}gx^- - \mathrm{i}gx^+}{2}}. \tag{3.62}$$

With this parametrization we find that the central charge H is expressed as

$$H = 1 + \frac{\mathrm{i}g}{x^+} - \frac{\mathrm{i}g}{x^-} = \mathrm{i}gx^- - \mathrm{i}gx^+ - 1, \tag{3.63}$$

while the remaining central charges take the form

$$C = \frac{\mathrm{i}g}{2}\left(\frac{x^+}{x^-} - 1\right)\mathrm{e}^{2\mathrm{i}\xi}, \qquad \overline{C} = \frac{g}{2\mathrm{i}}\left(\frac{x^-}{x^+} - 1\right)\mathrm{e}^{-2\mathrm{i}\xi}. \tag{3.64}$$

We will see that the *S*-matrix coefficients are conveniently expressed in terms of $x^{\pm}$.

In what follows we denote the fundamental representation as $\mathscr{V}(p,\zeta)$ (or just $\mathscr{V}$ if the values of p and ζ are not important), where $\zeta = \mathrm{e}^{2\mathrm{i}\xi}$.

3.2.4. Rapidity torus. Here we would like to find an analog of the rapidity variable for the non-Lorentz-invariant string sigma model and to understand the action of crossing symmetry.

Our starting point is the dispersion relation (3.56) for particles from the fundamental $\mathfrak{su}(2|2)_{\mathcal{C}}$-multiplet. This formula shows that the universal cover of the parameter space describing the representation is an elliptic curve. Indeed, equation (3.56) can be naturally uniformized in terms of Jacobi elliptic functions

$$p = 2\mathrm{am}z, \qquad \sin\frac{p}{2} = \mathrm{sn}(z,k), \qquad H = \mathrm{dn}(z,k), \tag{3.65}$$

where we introduced the elliptic modulus[26] $k = -4g^2 = -\frac{\lambda}{\pi^2} < 0$. The corresponding elliptic curve (the torus) has two periods $2\omega_1$ and $2\omega_2$, the first one is real and the second one is imaginary

$$2\omega_1 = 4\mathrm{K}(k), \qquad 2\omega_2 = 4\mathrm{i}\,\mathrm{K}(1-k) - 4\mathrm{K}(k),$$

where $\mathrm{K}(k)$ stands for the complete elliptic integral of the first kind. The dispersion relation is obviously invariant under shifts of z by $2\omega_1$ and $2\omega_2$. The torus parametrized by the complex variable z is the analog of the rapidity plane in two-dimensional relativistic models.

In this parametrization the real z-axis can be called the physical one for the original string theory, because for real values of z the energy is positive and the momentum is real due to

$$1 \leqslant \mathrm{dn}(z,k) \leqslant \sqrt{k'}, \qquad z \in \mathbb{R},$$

where $k' \equiv 1 - k$ is the complementary modulus.

We further note that the representation parameters $x^{\pm}$ are expressed in terms of Jacobi elliptic functions as

$$x^{\pm} = \frac{1}{2g}\left(\frac{\mathrm{cn}\,z}{\mathrm{sn}\,z} \pm \mathrm{i}\right)(1 + \mathrm{dn}\,z). \tag{3.66}$$

This form of $x^{\pm}$ follows from the requirement that for real values of z the absolute values of $x^{\pm}$ are greater than unity $|x^{\pm}| > 1$, and the imaginary parts satisfy $\mathrm{Im}(x^+) > 0$ and $\mathrm{Im}(x^-) < 0$.

The transformation properties of the parameters $x^{\pm}$ under shifts of z by some fractions of the periods are presented in the table 1. Since both the dispersion relation and $x^{\pm}$ are periodic with period ω_1, the range of the real part of z can be restricted to the interval from $-\omega_1/2$ to $\omega_1/2$ which corresponds to $-\pi \leqslant p \leqslant \pi$.

Now we analyze what happens to the torus in the limits $g \to \infty$ and $g \to 0$. When $g \to \infty$ the periods exhibit the following behavior:

$$\omega_1 \to \frac{\log g}{g}, \qquad \omega_2 \to \frac{\mathrm{i}\pi}{2g} \qquad \text{if} \quad g \to \infty. \tag{3.67}$$

To keep the range of $\mathrm{Im}(z)$ finite, we rescale z as $z \to z/(2g)$, and the momentum as $p \to p/g$. Then the dispersion relation (3.56) acquires the standard relativistic form (3.40), the variable

[26] Our convention for the elliptic modulus is the same as accepted in the *Mathematica* program, e.g., $\mathrm{sn}(z,k) = \mathrm{JacobiSN}[z,k]$. Throughout the paper we will often indicate only the z-dependence of Jacobi elliptic functions until it leads to any confusion.

Table 1. Transformations of $x^{\pm}$ under some shifts of z.

z	x^+	x^-
$z+\omega_1$	x^+	x^-
$z+\omega_2$	$1/x^+$	$1/x^-$
$z+\frac{1}{2}(\omega_1+\omega_2)$	$-1/x^-$	$-x^+$
$z+(\omega_1+\omega_2)$	$1/x^+$	$1/x^-$
$z+\frac{3}{2}(\omega_1+\omega_2)$	$-x^-$	$-1/x^+$
$-z$	$-x^-$	$-x^+$
$-z+\frac{1}{2}(\omega_1+\omega_2)$	$1/x^+$	x^-
$-z+(\omega_1+\omega_2)$	$-1/x^-$	$-1/x^+$
$-z+\frac{3}{2}(\omega_1+\omega_2)$	x^+	$1/x^-$

z plays the role of rapidity θ as $p = \sinh z$. As to the torus, it degenerates into a strip with $-\pi < \mathrm{Im}(z) < \pi$ and $-\infty < \mathrm{Re}(z) < \infty$. This is twice the physical strip of a relativistic field theory.

In the limit $g \to 0$ the periods of the torus have the following behavior:

$$\omega_1 \to \pi, \qquad \omega_2 \to 2\mathrm{i}\log g \qquad \text{if} \quad g \to 0. \tag{3.68}$$

Thus, the torus degenerates into the strip with $-\pi/2 < \mathrm{Re}(z) < \pi/2$ and $-\infty < \mathrm{Im}(z) < \infty$. The limit $g \to 0$ corresponds to the one-loop gauge theory.

An important property of our parametrization of the fundamental representation (3.61) is that if the parameter $\mathrm{e}^{\mathrm{i}\xi}$ is a meromorphic function on the torus then all the parameters a, b, c, d are meromorphic functions as well. To show this, one has to resolve the branch cut ambiguities arising from the parameter η (3.62).

This can be done in the following way. First, the elliptic parametrization (3.66) gives

$$\begin{aligned}\eta(p) &= \mathrm{e}^{\frac{\mathrm{i}}{4}p}\sqrt{\frac{\mathrm{i}gx^-(p) - \mathrm{i}gx^+(p)}{2}} = \frac{1}{\sqrt{2}}\,\mathrm{e}^{\frac{\mathrm{i}}{2}\,\mathrm{am}z}\sqrt{1+\mathrm{dn}\,z}\\ &= \frac{1}{\sqrt{2}}\sqrt{(1+\mathrm{dn}\,z)(\mathrm{cn}\,z + \mathrm{i}\,\mathrm{sn}z)}.\end{aligned} \tag{3.69}$$

Second, by using the following formulae (recall $k = -4g^2$)

$$1+\mathrm{dn}\,z = \frac{2\mathrm{dn}^2\frac{z}{2}}{1+4g^2\mathrm{sn}^4\frac{z}{2}}, \qquad \mathrm{cn}\,z + \mathrm{i}\,\mathrm{sn}z = \frac{\left(\mathrm{cn}\,\frac{z}{2} + \mathrm{i}\,\mathrm{sn}\frac{z}{2}\,\mathrm{dn}\,\frac{z}{2}\right)^2}{1+4g^2\mathrm{sn}^4\frac{z}{2}},$$

relating elliptic functions to those of the half argument, we can resolve the branch cut ambiguities by means of the relation

$$\mathrm{e}^{\frac{\mathrm{i}}{4}p}\sqrt{\frac{\mathrm{i}gx^-(p) - \mathrm{i}gx^+(p)}{2}} = \frac{\mathrm{dn}\,\frac{z}{2}\left(\mathrm{cn}\,\frac{z}{2} + \mathrm{i}\,\mathrm{sn}\frac{z}{2}\,\mathrm{dn}\,\frac{z}{2}\right)}{1+4g^2\mathrm{sn}^4\frac{z}{2}} \equiv \eta(z) \tag{3.70}$$

valid in the region $-\frac{\omega_1}{2} < \mathrm{Re}\,z < \frac{\omega_1}{2}$ and $-\omega_2/\,\mathrm{i} < \mathrm{Im}\,z < \omega_2/\,\mathrm{i}$. Finally, we note that since $\mathrm{e}^{-\frac{\mathrm{i}}{2}p} = \mathrm{cn}\,z - \mathrm{i}\,\mathrm{sn}z$, and $x^{\pm}$ are meromorphic functions, then the representation parameters a, b, c, d are meromorphic as well. This property greatly facilitates the treatment of crossing symmetry.

3.3. The $\mathfrak{su}(2|2)$-invariant S-matrix

Since the manifest symmetry algebra of the light-cone string theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$ consists of two copies of the centrally extended $\mathfrak{su}(2|2)$, the creation operators $A^{\dagger}_{M\dot{M}}(p)$ carry two indices

M and $\dot{M}$, where the dotted index is for the second $\mathfrak{su}(2|2)$. The n-particle states are obtained by acting with the creation operators on the vacuum

$$A^{\dagger}_{M_1\dot{M}_1}(p_1)\cdots A^{\dagger}_{M_n\dot{M}_n}(p_n)|\Omega\rangle \equiv \big|A^{\dagger}_{M_1\dot{M}_1}(p_1)\cdots A^{\dagger}_{M_n\dot{M}_n}(p_n)\big\rangle. \tag{3.71}$$

For the purpose of this section we can think of $A^{\dagger}_{M\dot{M}}(p)$ as being a product of two creation operators $A^{\dagger}_{M\dot{M}}(p) = A^{\dagger}_{M}(p) \times A^{\dagger}_{\dot{M}}(p)$ and restrict our attention to the states created by $A^{\dagger}_{M}(p)$.

3.3.1. Two-particle states and the S-matrix. It is clear that a one-particle state $\big|A^{\dagger}_{M}(p)\big\rangle$ is identified with the basis vector $|e_M\rangle$ of the fundamental representation $\mathscr{V}(p,1)$ of $\mathfrak{su}(2|2)_c$ (and we also set $\varphi = 0$). Let us stress that we have to set the parameter ζ to 1, because we use the canonical form of the central charge $\mathbb{C}$ with $\xi = 0$

$$\mathbb{C} = \frac{\mathrm{i}g}{2}(\mathrm{e}^{\mathrm{i}\mathbb{P}} - 1), \qquad \mathbb{C}\big|A^{\dagger}_{M}(p)\big\rangle = \frac{\mathrm{i}g}{2}(\mathrm{e}^{\mathrm{i}p} - 1)\big|A^{\dagger}_{M}(p)\big\rangle. \tag{3.72}$$

Then the two-particle states created by $A^{\dagger}_{M}(p)$ should be identified with the tensor product of fundamental representations of $\mathfrak{su}(2|2)_c$

$$\big|A^{\dagger}_{M_1}(p_1)A^{\dagger}_{M_2}(p_2)\big\rangle \sim \mathscr{V}(p_1,\zeta_1)\otimes\mathscr{V}(p_2,\zeta_2), \tag{3.73}$$

equipped with the canonical action of the symmetry generators in the tensor product. An important observation is that the parameters ζ_k cannot be equal to 1. The reason for that is very simple. Computing the central charge $\mathbb{C}$ on the two-particle state, we get

$$\mathbb{C}\big|A^{\dagger}_{M_1}(p_1)A^{\dagger}_{M_2}(p_2)\big\rangle = \frac{\mathrm{i}g}{2}(\mathrm{e}^{\mathrm{i}(p_1+p_2)} - 1)\big|A^{\dagger}_{M_1}(p_1)A^{\dagger}_{M_2}(p_2)\big\rangle, \tag{3.74}$$

because $\mathbb{P}A^{\dagger}_{M}(p) = A^{\dagger}_{M}(p)(\mathbb{P}+p)$. On the other hand, the value of the central charge on the tensor product of fundamental representations is equal to the sum of their charges

$$\mathbb{C}\mathscr{V}(p_1,\zeta_1)\otimes\mathscr{V}(p_2,\zeta_2) = \frac{\mathrm{i}g}{2}(\zeta_1(\mathrm{e}^{\mathrm{i}p_1}-1)+\zeta_2(\mathrm{e}^{\mathrm{i}p_2}-1))\mathscr{V}(p_1,\zeta_1)\otimes\mathscr{V}(p_2,\zeta_2).$$

Thus, we must have the following identity:

$$\mathrm{e}^{\mathrm{i}(p_1+p_2)} - 1 = \zeta_1(\mathrm{e}^{\mathrm{i}p_1}-1)+\zeta_2(\mathrm{e}^{\mathrm{i}p_2}-1), \tag{3.75}$$

which obviously cannot be satisfied if both ζ_1 and ζ_2 are equal to 1. In fact, it is easy to show that there are only two solutions to this equation for ζ_k lying on the unit circle

$$\{\zeta_1 = 1, \zeta_2 = \mathrm{e}^{\mathrm{i}p_1}\}, \qquad \text{or} \qquad \{\zeta_1 = \mathrm{e}^{\mathrm{i}p_2}, \zeta_2 = 1\}. \tag{3.76}$$

A priori any of these two solutions can be used to identify a two-particle state with the tensor product. However, the form of the S-matrix depends on the identification, and, as we will see shortly, it is the first solution that leads to the S-matrix which precisely agrees with the perturbative S-matrix discussed in the previous section.

It is readily seen that the first solution corresponds to the following rearrangement of the commutation relation of the central charge $\mathbb{C}$ with $A^{\dagger}_{M}(p)$

$$\mathbb{C}A^{\dagger}_{M}(p) = C(p)A^{\dagger}_{M}(p) + \mathrm{e}^{\mathrm{i}p}A^{\dagger}_{M}(p)\mathbb{C}, \tag{3.77}$$

while the second solution corresponds to another rearrangement of the commutation relation

$$\mathbb{C}A^{\dagger}_{M}(p) = C(p)A^{\dagger}_{M}(p)\,\mathrm{e}^{\mathrm{i}\mathbb{P}} + A^{\dagger}_{M}(p)\mathbb{C}. \tag{3.78}$$

The latter has an explicit dependence on the operator of the world-sheet momentum.

Thus, taking the first solution in (3.76), we see that the invariance condition (3.25) takes the following form for bosonic generators ${L_a}^b$ and ${R_\alpha}^\beta$

$$S_{12}(p_1, p_2)(J \otimes \mathbb{1} + \mathbb{1} \otimes J) = (J \otimes \mathbb{1} + \mathbb{1} \otimes J)S_{12}(p_1, p_2), \tag{3.79}$$

and for fermionic generators ${Q_\alpha}^a$ and $Q_a^{\dagger\alpha}$

$$\begin{aligned} &S_{12}(p_1, p_2)(J(p_1; 1) \otimes \mathbb{1} + \Sigma \otimes J(p_2; \mathrm{e}^{\mathrm{i}p_1})) \\ &\qquad = (J(p_1; \mathrm{e}^{\mathrm{i}p_2}) \otimes \Sigma + \mathbb{1} \otimes J(p_2; 1))S_{12}(p_1, p_2), \end{aligned} \tag{3.80}$$

where $J(p; \zeta)$ denote the structure constants matrices of the fundamental representation parametrized by g, p and $\zeta = \mathrm{e}^{2\mathrm{i}\xi}$, see (3.50) and (3.61). The grading matrix

$$\Sigma = \mathrm{diag}(1, 1, -1, -1) \tag{3.81}$$

defined in equation (3.14) takes care of the negative sign for fermions. These are the conditions that should be used to find the *S*-matrix. With the choice of the representation parameters we made, the resulting *S*-matrix satisfies the Yang–Baxter equation.

We note that if we think of vectors from $\mathscr{V}(p; \zeta)$ as columns then, as is seen from equation (3.80), the *S*-matrix can be considered as a map

$$S_{12}(p_1, p_2) : \mathscr{V}(p_1, 1) \otimes \mathscr{V}(p_2, \mathrm{e}^{\mathrm{i}p_1}) \to \mathscr{V}(p_1, \mathrm{e}^{\mathrm{i}p_2}) \otimes \mathscr{V}(p_2, 1), \tag{3.82}$$

and if we think of vectors from $\mathscr{V}(p; \zeta)$ as rows then the *S*-matrix can be regarded as the opposite map

$$S_{12}(p_1, p_2) : \mathscr{V}(p_1, \mathrm{e}^{\mathrm{i}p_2}) \otimes \mathscr{V}(p_2, 1) \to \mathscr{V}(p_1, 1) \otimes \mathscr{V}(p_2, \mathrm{e}^{\mathrm{i}p_1}). \tag{3.83}$$

From this point of view the action of the *S*-matrix corresponds to exchanging the two possible choices of the parameters ζ_k of the two representations. Let us stress, however, that no matter what interpretation we use, $S_{12}(p_1, p_2)$ is a 16×16 matrix acting in the 16-dimensional vector space of the two-particle states $\big|A^\dagger_{M_2}(p_2)A^\dagger_{M_1}(p_1)\big\rangle$.

When the string coupling constant g tends to infinity, the string sigma-model becomes free, and the ZF creation operators turn into the usual creation operators, i.e. commute or anti-commute depending on the statistics. Therefore, in this limit the *S*-matrix should be equal to the graded unity.

The *S*-matrix satisfying equations (3.79) and (3.80) can be easily found up to an overall scalar factor. In the following we will give up the particle momenta $p_i = 2\,\mathrm{am}\, z_i$ in favor of the rapidity variables z_i. The invariance condition (3.79) that involves the bosonic algebra generators fixes the form of the *S*-matrix up to ten arbitrary coefficients

$$S(z_1, z_2) = \sum_{k=1}^{10} a_k \Lambda_k, \tag{3.84}$$

where $\Lambda_1, \ldots, \Lambda_{10}$ form a basis of $\mathfrak{su}(2) \otimes \mathfrak{su}(2)$ invariant matrices acting in the tensor product $\mathscr{V}(z_1) \otimes \mathscr{V}(z_2)$

$$\Lambda_1 = E_{1111} + \tfrac{1}{2}E_{1122} + \tfrac{1}{2}E_{1221} + \tfrac{1}{2}E_{2112} + \tfrac{1}{2}E_{2211} + E_{2222},$$

$$\Lambda_2 = \tfrac{1}{2}E_{1122} - \tfrac{1}{2}E_{1221} - \tfrac{1}{2}E_{2112} + \tfrac{1}{2}E_{2211},$$

$$\Lambda_3 = E_{3333} + \tfrac{1}{2}E_{3344} + \tfrac{1}{2}E_{3443} + \tfrac{1}{2}E_{4334} + \tfrac{1}{2}E_{4433} + E_{4444},$$

$$\Lambda_4 = \tfrac{1}{2}E_{3344} - \tfrac{1}{2}E_{3443} - \tfrac{1}{2}E_{4334} + \tfrac{1}{2}E_{4433},$$

$$\Lambda_5 = E_{1133} + E_{1144} + E_{2233} + E_{2244},$$

$$\Lambda_6 = E_{3311} + E_{3322} + E_{4411} + E_{4422},$$

$$\Lambda_7 = E_{1324} - E_{1423} - E_{2314} + E_{2413},$$

$$\Lambda_8 = E_{3142} - E_{3241} - E_{4132} + E_{4231},$$

$$\Lambda_9 = E_{1331} + E_{1441} + E_{2332} + E_{2442},$$

$$\Lambda_{10} = E_{3113} + E_{3223} + E_{4114} + E_{4224}.$$

Here the symbols E_{kilj} are equal to $(-1)^{\epsilon_k\epsilon_l} E_k{}^i \otimes E_l{}^j$, where $E_k{}^i \equiv E_{ki}$ are the standard 4×4 matrix unities[27]. The normalization of Λ_i has been chosen in such a way that for $a_1 = a_2 = \cdots = a_6 = 1$ and $a_7 = a_8 = a_9 = a_{10} = 0$ the matrix S coincides with the graded identity.

The unknown coefficients a_k can be now determined from the permutation relations of the S-matrix with the supersymmetry generators. We find

$$\begin{aligned}
a_1 &= 1,\\
a_2 &= 2\frac{(x_1^- - x_2^-)\left(x_1^+x_2^- - 1\right)x_2^+}{\left(x_1^- - x_2^+\right)\left(x_1^+x_2^+ - 1\right)x_2^-} - 1,\\
a_3 &= \frac{x_1^+ - x_2^-}{x_1^- - x_2^+}\frac{\tilde\eta_1\tilde\eta_2}{\eta_1\eta_2},\\
a_4 &= -\frac{x_1^+ - x_2^-}{x_1^- - x_2^+}\frac{\tilde\eta_1\tilde\eta_2}{\eta_1\eta_2} + 2\frac{(x_1^- - x_2^-)(x_1^-x_2^+ - 1)x_1^+}{(x_1^- - x_2^+)\left(x_1^+x_2^+ - 1\right)x_1^-}\frac{\tilde\eta_1\tilde\eta_2}{\eta_1\eta_2},\\
a_5 &= \frac{x_1^+ - x_2^+}{x_1^- - x_2^+}\frac{\tilde\eta_2}{\eta_2},\\
a_6 &= \frac{x_1^- - x_2^-}{x_1^- - x_2^+}\frac{\tilde\eta_1}{\eta_1},\\
a_7 &= \frac{g}{2\mathrm{i}}\frac{\left(x_1^- - x_1^+\right)\left(x_1^+ - x_2^+\right)\left(x_2^- - x_2^+\right)}{\left(x_1^- - x_2^+\right)\left(x_1^-x_2^- - 1\right)}\frac{1}{\eta_1\eta_2},\\
a_8 &= \frac{2\mathrm{i}}{g}\frac{(x_1^- - x_2^-)}{\left(x_1^- - x_2^+\right)\left(x_1^+x_2^+ - 1\right)}\tilde\eta_1\tilde\eta_2,\\
a_9 &= \frac{x_1^- - x_1^+}{x_1^- - x_2^+}\frac{\tilde\eta_2}{\eta_1},\\
a_{10} &= \frac{x_2^- - x_2^+}{x_1^- - x_2^+}\frac{\tilde\eta_1}{\eta_2}.
\end{aligned}$$

The coefficients a_k are determined up to an overall scaling factor, and we normalize them in a canonical way by setting $a_1 = 1$. The parameters η_k are not fixed by the invariance condition. They are determined by imposing the generalized unitarity condition and the Yang–Baxter equation, and are given by the following formulae:

$$\begin{aligned}
\eta_1 &= \eta(z_1), & \tilde\eta_1 &= (\mathrm{cn}\, z_2 + \mathrm{i}\,\mathrm{sn}\, z_2)\eta(z_1)\\
\tilde\eta_2 &= \eta(z_2), & \tilde\eta_2 &= (\mathrm{cn}\, z_1 + \mathrm{i}\,\mathrm{sn}\, z_1)\eta(z_2),
\end{aligned} \tag{3.85}$$

where $\eta(z)$ is defined by (3.70).

[27] Choosing $E_{kilj} \equiv E_k{}^i \otimes E_l{}^j$ will produce the corresponding graded S-matrix S^g.

An important property of the *S*-matrix (3.84) is that up to the scalar factor it is a meromorphic function of the torus variables z_1, z_2 because the parameters of all the four representations appearing in the invariance condition (3.80) are meromorphic. In what follows we often refer to the *S*-matrix (3.84) with the coefficients a_i given above as to the canonical $\mathfrak{su}(2|2)$-invariant fundamental *S*-matrix.

The canonical *S*-matrix (3.84) satisfies all the properties we discussed in subsection 3.1.3. First, the physical unitarity condition $S(z_1, z_2)^\dagger \cdot S(z_1, z_2) = \mathbb{1}$ for z_1, z_2 real can be easily checked by using the explicit form of the coefficients a_i, and the Hermitian conjugation and transposition conditions

$$(\Lambda_i)^\dagger = (\Lambda_i)^t = \Lambda_i, \quad i = 1, \dots, 6; \quad (\Lambda_7)^\dagger = (\Lambda_7)^t = -\Lambda_8, \quad (\Lambda_9)^\dagger = (\Lambda_9)^t = \Lambda_{10}.$$

Moreover, with the choice (3.85) of η_i, the *S*-matrix also satisfies the generalized unitarity condition $S(z_1^*, z_2^*)^\dagger \cdot S(z_1, z_2) = \mathbb{1}$, and it is also a graded-symmetric matrix $S^t(z_1, z_2) = \mathbb{1}^g S(z_1, z_2)\mathbb{1}^g$. The latter property implies that the coefficients a_i satisfy the following relations:

$$a_7(z_1, z_2) = a_8(z_1, z_2), \qquad a_9(z_1, z_2) = a_{10}(z_1, z_2), \tag{3.86}$$

which, in fact, reduces the number of independent coefficients to seven.

If $p_1 = p_2$ or, equivalently, $z_1 = z_2$ the canonical *S*-matrix becomes the permutation matrix. As will become clear in the following section, the world-sheet *S*-matrix reduces at this special point to minus the permutation due to the scalar factor which tends to -1.

We further note that the form of the structure constants matrices $J(p; \zeta)$ allows one to determine the commutation relations of the symmetry operators with the creation and annihilation operators. It is convenient to use the matrix notations, i.e. to combine $A_M^\dagger$ and A_M into a row and column, respectively, and the symmetry algebra structure constants of the one-particle representation (3.61) with $\xi = 0$ into matrices $L_a{}^b$, $R_\alpha{}^\beta$, $Q_\alpha{}^a$ and $Q_a^{\dagger\alpha}$, see (3.49). Then, the commutation relations for the centrally extended algebra $\mathfrak{su}(2|2)_{\mathcal{C}}$ can be written in the following simple form of the braided (anti)-commutators:

$$\begin{aligned}
&\mathbb{L}_a{}^b \mathbb{A}^\dagger(p) - \mathbb{A}^\dagger(p)\mathbb{L}_a{}^b = \mathbb{A}^\dagger(p) L_a{}^b,\\
&\mathbb{R}_\alpha{}^\beta \mathbb{A}^\dagger(p) - \mathbb{A}^\dagger(p)\mathbb{R}_\alpha{}^\beta = \mathbb{A}^\dagger(p) R_\alpha{}^\beta,\\
&\mathbb{Q}_\alpha{}^a \mathbb{A}^\dagger(p) - \mathrm{e}^{\mathrm{i}p/2}\mathbb{A}^\dagger(p)\Sigma\mathbb{Q}_\alpha{}^a = \mathbb{A}^\dagger(p) Q_\alpha{}^a(p),\\
&\mathbb{Q}_a^{\dagger\alpha} \mathbb{A}^\dagger(p) - \mathrm{e}^{-\mathrm{i}p/2}\mathbb{A}^\dagger(p)\Sigma\mathbb{Q}_a^{\dagger\alpha} = \mathbb{A}^\dagger(p) Q_a^{\dagger\alpha}(p).
\end{aligned} \tag{3.87}$$

Thus, the braiding factors in (3.87) are the exponents $\mathrm{e}^{\pm \mathrm{i}p/2}$. This form of the commutation relations is the one that usually appears in models with non-local charges.

It is worthwhile to note that the form of the two-particle structure constant matrices appearing in the invariance condition (3.80) allows us to reformulate the problem by using the Hopf algebra language, see appendix 3.5.3 for detail.

3.3.2. Multi-particle states. Multi-particle states created by $A_M^\dagger(p)$ are correspondingly identified with the tensor product of fundamental representations of $\mathfrak{su}(2|2)_{\mathcal{C}}$

$$\left|A_{M_1}^\dagger(p_1)\cdots A_{M_n}^\dagger(p_n)\right\rangle \sim \mathscr{V}(p_1, \zeta_1) \otimes \cdots \otimes \mathscr{V}(p_n, \zeta_n), \tag{3.88}$$

equipped with the canonical action of the symmetry generators in the tensor product, and the parameters ζ_k have to satisfy the following identity:

$$\mathrm{e}^{\mathrm{i}(p_1+\cdots+p_n)} - 1 = \sum_{k=1}^{n} \zeta_k(\mathrm{e}^{\mathrm{i}p_k} - 1). \tag{3.89}$$

In general, there are many different solutions to this equation. In our case, however, the choice of ζ_k is fixed by the commutation relations (3.87) and (3.77). One can easily see that the only solution compatible with (3.87) is

$$\zeta_1 = 1, \qquad \zeta_2 = e^{ip_1}, \ldots, \qquad \zeta_{n-1} = e^{i(p_1+\cdots+p_{n-2})}, \qquad \zeta_n = e^{i(p_1+\cdots+p_{n-1})}. \tag{3.90}$$

It is clear that the multi-particle S-matrix just maps the vector space with this choice of ζ_k to the (isomorphic) space with the following choice of ζ_k:

$$\zeta_1 = e^{i(p_2+\cdots+p_n)}, \qquad \zeta_2 = e^{i(p_3+\cdots+p_n)}, \ldots, \qquad \zeta_{n-1} = e^{ip_n}, \qquad \zeta_n = 1, \tag{3.91}$$

because the second choice obviously corresponds to the order of the ZF creation operators in the *out*-state.

Due to the integrability of the model the multi-particle S-matrix factorizes into a product of two-particle ones, and the consistency condition for the factorizability is equivalent to the Yang–Baxter equation.

3.4. Crossing symmetry

3.4.1. World-sheet S-matrix and dressing phase. The canonical $\mathfrak{su}(2|2)$-invariant fundamental S-matrix can be used to find the corresponding $\mathfrak{su}(2|2) \oplus \mathfrak{su}(2|2)$ invariant world-sheet S-matrix which describes the scattering of fundamental particles in the light-cone string sigma model. To this end, one should multiply the tensor product of two copies of the canonical S-matrix by a scalar factor so that the resulting matrix would satisfy an equation imposed by crossing symmetry. Thus, the world-sheet S-matrix describing the scattering of two fundamental particles is of the form

$$\mathcal{S}(z_1, z_2) = S_0(z_1, z_2) S(z_1, z_2) \check{\otimes} S(z_1, z_2). \tag{3.92}$$

The tensor product in (3.92) is unusual and it takes care of various signs which arise due to factorization of the ZF creation operators. For the graded S-matrix these signs were determined in subsection 2.3.3, see equation (2.110). Taking into account that the graded S-matrix is equal to the product of the graded identity and the S-matrix (3.92), one finds the indexed version of equation (3.92)

$$\mathcal{S}_{M\dot{M},N\dot{N}}^{P\dot{P},Q\dot{Q}}(z_1, z_2) = (-1)^{\epsilon_{\dot{M}}\epsilon_N + \epsilon_P\epsilon_{\dot{Q}}} S_0(z_1, z_2) S_{MN}^{PQ}(z_1, z_2) \dot{S}_{\dot{M}\dot{N}}^{\dot{P}\dot{Q}}(z_1, z_2). \tag{3.93}$$

Since Λ_1 is the only $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ invariant matrix which contains the term $E_1{}^1 \otimes E_1{}^1$, the S-matrix component

$$\mathcal{S}_{1\dot{1},1\dot{1}}^{1\dot{1},1\dot{1}}(z_1, z_2) \equiv S_{\mathfrak{su}(2)}(z_1, z_2) = S_0(z_1, z_2) a_1(z_1, z_2)^2$$

describes the scattering of particles in the $\mathfrak{su}(2)$ sector of the theory. Since we have set a_1 equal to unity, the scalar factor S_0 in equation (3.92) is simply equal to the S-matrix of the $\mathfrak{su}(2)$ sector

$$S_0(z_1, z_2) = S_{\mathfrak{su}(2)}(z_1, z_2).$$

Thus, assuming integrability and preservation of classical symmetries at the quantum level, we conclude that the S-matrix in the $\mathfrak{su}(2)$ sector encodes the full dynamics of the model as its form cannot be fixed by kinematical symmetries. This S-matrix was determined by using various indirect arguments involving both string and gauge theory considerations which will be discussed in part II of the review. Here we will present the resulting expression

$$S_{\mathfrak{su}(2)}(z_1, z_2) = e^{ia(p_2\omega_1 - p_1\omega_2)} \frac{x_1^+}{x_1^-} \frac{x_2^-}{x_2^+} \frac{1}{\sigma\left(x_1^\pm, x_2^\pm\right)^2} \frac{u_1 - u_2 - \frac{2i}{g}}{u_1 - u_2 + \frac{2i}{g}}. \tag{3.94}$$

In equation (3.94) the spectral parameters u_k are expressed in terms of $x_k^{\pm}$ as follows

$$u_k = \frac{1}{2}\left(x_k^+ + \frac{1}{x_k^+} + x_k^- + \frac{1}{x_k^-}\right),$$

and in terms of the u-parameters the last term in (3.94) is nothing else but the S-matrix of the Heisenberg spin chain. It exhibits a pole at $u_1 - u_2 = -\frac{2i}{g}$ which corresponds to a bound state of two fundamental particles from the $\mathfrak{su}(2)$ sector, as we will show in part II.

The first factor in (3.94) depends on a which is the parameter of the uniform light-cone gauge (2.8), and

$$\omega_i = \sqrt{1 + 4g^2 \sin^2 \frac{p_i}{2}}$$

is the energy of the ith particle. Under crossing both p and ω change sign and, as a consequence, the gauge-dependent factor solves the homogeneous crossing equation. Without loss of generality, in what follows we set $a = 0$.

The gauge-independent function $\sigma\left(x_1^{\pm}, x_2^{\pm}\right)$ is called the dressing factor, and it is often written in the exponential form $\sigma\left(x_1^{\pm}, x_2^{\pm}\right) = e^{i\theta(x_1^{\pm}, x_2^{\pm})}$. Here the dressing phase

$$\theta(x_1^+, x_1^-, x_2^+, x_2^-) = \sum_{r=2}^{\infty} \sum_{\substack{s>r \\ r+s=\text{odd}}}^{\infty} c_{r,s}(g)\left[q_r\left(x_1^{\pm}\right)q_s\left(x_2^{\pm}\right) - q_r\left(x_2^{\pm}\right)q_s\left(x_1^{\pm}\right)\right] \tag{3.95}$$

is a 2-form on the vector space of conserved charges $q_r(x^{\pm})$

$$q_r(x_k^-, x_k^+) = \frac{i}{r-1}\left[\left(\frac{1}{x_k^+}\right)^{r-1} - \left(\frac{1}{x_k^-}\right)^{r-1}\right]. \tag{3.96}$$

The coefficients $c_{r,s}(g)$ are non-trivial real functions of the string tension and they admit an asymptotic large g expansion

$$c_{r,s}(g) = g \sum_{n=0}^{\infty} \frac{1}{g^n} c_{r,s}^{(n)}, \qquad g \gg 1, \tag{3.97}$$

where the numerical coefficients $c_{r,s}^{(n)}$ can be determined from string sigma model perturbative computations. The leading order coefficients $c_{r,s}^{(0)}$ and the functional form (3.95) of the dressing phase were found by discretizing the finite-gap integral equations which describe the spectrum of classical spinning strings. The result is

$$c_{r,s}^{(0)} = \tfrac{1}{2}\delta_{r+1,s}. \tag{3.98}$$

The leading coefficients (3.98) are already enough to relate the exact world-sheet S-matrix we discuss here and the tree-level S-matrix computed in section 2. First, we construct the graded version of the exact S-matrix: $\mathbb{S}^g(p_1, p_2) = \mathbb{1}^g \mathcal{S}(p_1, p_2)$. Second, we rescale the particle momenta $p_i \to p_i/g$ and take the limit $g \to \infty$. One then finds that the leading term in the strong coupling expansion of the exact (graded) world-sheet S-matrix is the identity, while the subleading one reproduces precisely the perturbative S-matrix of section 2.

Returning to the discussion of the dressing phase, the subleading coefficients in (3.97) were fixed by analyzing the one-loop corrections to energies of circular spinning strings, and turn out to be

$$c_{r,s}^{(1)} = -\frac{2}{\pi}\frac{(r-1)(s-1)}{(r+s-2)(s-r)}, \qquad r+s = \text{odd}. \tag{3.99}$$

The requirement that the dressing factor satisfies the crossing symmetry equations leads to the following proposal for all the coefficients $c_{r,s}^{(n)}$:

$$c_{r,s}^{(n)} = \frac{(-1)^n\zeta(n)}{2\pi^n\Gamma(n-1)}(r-1)(s-1)\frac{\Gamma\left[\frac{1}{2}(s+r+n-3)\right]\Gamma\left[\frac{1}{2}(s-r+n-1)\right]}{\Gamma\left[\frac{1}{2}(s+r-n+1)\Gamma\left[\frac{1}{2}(s-r-n+3)\right]\right.}, \tag{3.100}$$

where we use that $r+s=$ odd.

The coefficients $c_{r,s}(g)$ also admit a convergent small g expansion

$$c_{r,s}(g) = g\sum_{n=r+s-3}^{\infty} g^n\tilde{c}_{r,s}^{(n)}, \qquad g<\frac{1}{2}. \tag{3.101}$$

where the numerical coefficients $\tilde{c}_{r,s}^{(n)}$ can, in principle, be extracted from anomalous dimensions of primary operators of the perturbative gauge theory. The first nonvanishing coefficient $\tilde{c}_{2,3}^{(2)}$ requires an involved four-loop perturbative computation and it appears to be

$$\tilde{c}_{2,3}^{(2)} = -\frac{\zeta(3)}{2}. \tag{3.102}$$

The remaining coefficients were conjectured by assuming analytic continuation, and are given by

$$\tilde{c}_{r,s}^{(n)} = \frac{\cos\left(\frac{1}{2}\pi n\right)(-1)^{s+n}2^{-n}\zeta(1+n)\Gamma(2+n)\Gamma(1+n)(r-1)(s-1)}{\Gamma\left[\frac{5+n-r-s}{2}\right]\Gamma\left[\frac{3+n+r-s}{2}\right]\Gamma\left[\frac{3+n-r+s}{2}\right]\Gamma\left[\frac{1+n+r+s}{2}\right]}, \tag{3.103}$$

where we use again that $r+s=$ odd. This formula shows that the coefficients are nonvanishing for, and only for, even n.

3.4.2. Crossing equations. Here we will come back to the issue of crossing symmetry, which is essentially related to the existence of the two branches of the dispersion relation, the one corresponding to unitary representations with $H>0$ and the other corresponding to anti-unitary ones with $H<0$.

Recall that on the upper sheet of the hyperboloid (3.48) the variable z takes values $-\omega_1/2\leqslant z\leqslant\omega_1/2$. Shifting z by half of the imaginary period, we find

$$\begin{aligned} H(z) &\to H(z+\omega_2) = \mathrm{dn}(z+\omega_2,k) = -\,\mathrm{dn}(z,k) = -H(z),\\ p(z) &\to p(z+\omega_2) = 2\mathrm{am}(z+\omega_2) = -2\mathrm{am}(z) = -p(z). \end{aligned} \tag{3.104}$$

Thus, under this transformation the positive energy branch of the dispersion relation transforms into the negative one; both the Hamiltonian and the momentum change their sign. Thus, the map $z\to z+\omega_2$ is the analog of the crossing symmetry transformation in two-dimensional relativistic field theories. In what follows we regard z as a complex variable and refer to equation (3.104) as the crossing transform.

Let $M\equiv M(H,C)$ be a matrix realization of a fundamental unitary irrep of $\mathfrak{su}(2|2)_c$ characterized by the central charge values H and C. Consider now the following map ('minus supertransposition'):

$$M\to -M^{\mathrm{st}}.$$

Obviously, under this map the central charge values change their signs. Moreover, $-M(H,C)^{\mathrm{st}}$ is an irrep of $\mathfrak{su}(2|2)_c$, but with exactly the opposite values of the central charges. In particular, if $M(H,C)$ belongs to the positive branch of the dispersion relation, then $-M(H,C)^{\mathrm{st}}$ is on the negative branch.

There are two transformations acting on the space of central charges: the first one is the crossing transform which essentially interchanges the positive and negative sheets between themselves, the second one is an outer automorphism belonging to SU(1, 1). By combining these two, one is always able to transform $(-H, -\mathrm{C})$ into (H, C). To understand this issue, we recall that in the elliptic parametrization both H and C are functions of z. Under the crossing transform, the Hamiltonian and the momentum change sign, which is, however, not always the case for $C(z)$. In fact, we find that

$$C(z+\omega_2) = \frac{\mathrm{i}}{2}g(\mathrm{e}^{-\mathrm{i}p}-1)\,\mathrm{e}^{2\mathrm{i}\xi} = -\mathrm{e}^{-\mathrm{i}p}C(z),$$

where we assume that ξ is independent of z.

On the other hand, one should recall a U(1) automorphism (a part of the outer SU(1, 1) automorphism group), which acts on the super- and central charges as

$$Q(z) \to \mathrm{e}^{\mathrm{i}\rho}Q(z), \qquad C(z) \to \mathrm{e}^{2\mathrm{i}\rho}C(z).$$

Thus, if we pick the U(1)-automorphism obeying the condition

$$\mathrm{e}^{2\mathrm{i}\rho}C(z+\omega_2) = -\mathrm{e}^{2\mathrm{i}\rho}\,\mathrm{e}^{-\mathrm{i}p}C(z) = -C(z),$$

i.e., $\mathrm{e}^{\mathrm{i}\rho} = \mathrm{e}^{\mathrm{i}\frac{p}{2}}$, then after applying the combined crossing and U(1) transformations, an original irrep with $(-H, -C)$ will receive the central charges (H, C) and, for this reason, must be equivalent to $M(H, C)$. In other words,

$$-\,\hat{\rho}(M(z+\omega_2))^{\mathrm{st}} = \mathscr{C}M(z)\mathscr{C}^{-1}, \tag{3.105}$$

where $\mathscr{C}$ is an intertwining matrix and $\hat{\rho}$ denotes the action of the U(1)-automorphism. In particular, specifying equation (3.105) for the kinematical generators we get

$$\mathscr{C}L_a^{\ b} = -L_b^{\ a}\mathscr{C}, \qquad \mathscr{C}R_\alpha^{\ \beta} = -R_\beta^{\ \alpha}\mathscr{C}, \tag{3.106}$$

where we have taken into account that $\left(L_a^{\ b}\right)^t = L_b^{\ a}$, and $\left(R_\alpha^{\ \beta}\right)^t = R_\beta^{\ \alpha}$. These relations fix the form of $\mathscr{C}$ up to two coefficients

$$\mathscr{C} = \begin{pmatrix} c_1\sigma_2 & 0 \\ 0 & c_2\sigma_2 \end{pmatrix},$$

where σ_2 is the Pauli matrix. It is clear that only the ratio c_1/c_2 matters, and in what follows we set $c_1 = 1$ for definiteness. Then, specification of equation (3.105) for the supersymmetry generators gives

$$\begin{aligned} &\mathrm{e}^{\mathrm{i}\frac{p}{2}}\,Q_\alpha^{\ a}(z+\omega_2)^{\mathrm{st}} = -\mathscr{C}Q_\alpha^{\ a}(z)\mathscr{C}^{-1}, \\ &\mathrm{e}^{-\mathrm{i}\frac{p}{2}}\,Q_a^{\dagger\alpha}(z+\omega_2)^{\mathrm{st}} = -\mathscr{C}Q_a^{\dagger\alpha}(z)\mathscr{C}^{-1}. \end{aligned} \tag{3.107}$$

The transformed supersymmetry generators can be easily found by using the following relations:

$$x^\pm(z+\omega_2) = 1/x^\pm(z), \qquad \eta(z+\omega_2) = \frac{\mathrm{i}}{x^+(z)}\eta(z). \tag{3.108}$$

One can further show that the matrix $\mathscr{C}$ is given by[28]

$$\mathscr{C} = \begin{pmatrix} \sigma_2 & 0 \\ 0 & \mathrm{i}\sigma_2 \end{pmatrix}. \tag{3.109}$$

[28] Essentially, $\mathscr{C}$ is a product of the charge conjugation and the parity transform matrices:

$$\mathscr{C} = -\,\mathrm{i}^{1/2}\begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix}\begin{pmatrix} \mathrm{i}^{1/2}\mathbb{1}_2 & 0 \\ 0 & \mathrm{i}^{-1/2}\mathbb{1}_2 \end{pmatrix},$$

where ϵ is defined in equation (1.131).

It is worthwhile mentioning that the representation obtained by shifting z in the opposite direction is related to the original one through the matrix $\mathscr{C}^{-1}$

$$\begin{aligned} e^{i\frac{p}{2}} Q_\alpha{}^a(z-\omega_2)^{st} &= -\mathscr{C}^{-1} Q_\alpha{}^a(z)\mathscr{C}, \\ e^{-i\frac{p}{2}} Q_a^{\dagger\alpha}(z-\omega_2)^{st} &= -\mathscr{C}^{-1} Q_a^{\dagger\alpha}(z)\mathscr{C}, \end{aligned} \tag{3.110}$$

because $\eta(z-\omega_2) = -\frac{i}{x^+(z)}\eta(z)$.

To derive the crossing equations, we use equations (3.107), (3.110) that relate the contragradient representation to the original one, and the invariance conditions (3.79), (3.80). Taking the transpose of (3.79) with respect to the first factor in the tensor product of two matrices, and using the relations (3.106), we get that the matrix $\mathscr{C}_1^{-1} S_{12}^{t_1} \mathscr{C}_1$ is $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$-invariant, i.e. it commutes with the bosonic generators.

Next, we rewrite equation (3.80) in the following form:

$$S_{12}(z_1,z_2)[J_1(z_1;1) + \Sigma_1 J_2(z_2; e^{ip(z_1)})] = [J_1(z_1; e^{ip(z_2)})\Sigma_2 + J_2(z_2;1)]S_{12}(z_1,z_2), \tag{3.111}$$

where the subscripts 1, 2 indicate the embedding of the matrices into the tensor product: $J_1(z;\zeta) \equiv J(z;\zeta)\otimes \mathbb{1}$, $J_2(z;\zeta) \equiv \mathbb{1}\otimes J(z;\zeta)$. Then, we take the transpose of equation (3.111) with respect to the first factor in the tensor product of two matrix spaces, and use the relations (3.110), (3.107) written in the form[29]

$$\begin{aligned} Q_\alpha{}^a(z;\zeta)^t &= -e^{\frac{ip(z)}{2}} \mathscr{C} Q_\alpha{}^a(z-\omega_2;\zeta)\mathscr{C}^{-1}\Sigma, \\ Q_\alpha{}^a(z;\zeta)^t &= -e^{\frac{ip(z)}{2}} \mathscr{C}^{-1} Q_\alpha{}^a(z+\omega_2;\zeta)\mathscr{C}\Sigma, \end{aligned} \tag{3.112}$$

and similar formulae for $Q_a^{\dagger\alpha}$. By using the first formula in (3.112), after a simple computation, we find that the matrix $\mathscr{C}_1 S_{12}^{t_1}(z_1+\omega_2, z_2)\mathscr{C}_1^{-1}$ satisfies the same invariance conditions as the matrix $S_{12}^{-1}(z_1,z_2)$ and, therefore, the two matrices can differ only by a function of z_1, z_2. The crossing symmetry condition is just a statement that an $\mathfrak{su}(2|2)$-invariant S-matrix could be multiplied by a scalar factor such that these two matrices become equal to each other

$$\mathscr{C}_1 S_{12}^{t_1}(z_1+\omega_2, z_2)\mathscr{C}_1^{-1} = S_{12}^{-1}(z_1,z_2). \tag{3.113}$$

In the same way, transposing equation (3.111) with respect to the second factor, we derive the second crossing equation

$$\mathscr{C}_2 S_{12}^{t_2}(z_1, z_2-\omega_2)\mathscr{C}_2^{-1} = S_{12}^{-1}(z_1,z_2). \tag{3.114}$$

The crossing equations impose important restrictions on the form of the S-matrix scalar factor. We find, in particular, that the S-matrix in the $\mathfrak{su}(2)$ sector should satisfy the following crossing symmetry equations:

$$\begin{aligned} S_{\mathfrak{su}(2)}(z_1,z_2)S_{\mathfrak{su}(2)}(z_1+\omega_2,z_2) &= f\left(x_1^\pm, x_2^\pm\right)^2, \\ S_{\mathfrak{su}(2)}(z_1,z_2)S_{\mathfrak{su}(2)}(z_1,z_2-\omega_2) &= f\left(x_1^\pm, x_2^\pm\right)^2, \end{aligned} \tag{3.115}$$

where the function $f\left(x_1^\pm, x_2^\pm\right)$ is defined by

$$f\left(x_1^\pm, x_2^\pm\right) = \frac{(x_1^- - x_2^-)\left(1-\frac{1}{x_1^- x_2^+}\right)}{\left(x_1^+ - x_2^-\right)\left(1-\frac{1}{x_1^+ x_2^+}\right)} \tag{3.116}$$

where the variables $x_i^\pm$ should be expressed through z_i by using equation (3.66).

[29] Here we have taken into account that $M^{st} = M^t\Sigma$ and also indicated a possible dependence of the supersymmetry generators on the parameter ζ.

These equations together with formula (3.94) can be used to derive the crossing equations for the dressing factor σ. In fact, the simplest form of these equations arises for a function $\Sigma(z_1, z_2)$ which differs from σ by the extra factor $\frac{x_1^-}{x_1^+}\frac{x_2^+}{x_2^-}$ entering in equation (3.94)

$$\Sigma(z_1, z_2) = \left(\frac{x_1^-}{x_1^+}\frac{x_2^+}{x_2^-}\right)^{\frac{1}{2}} \sigma\left(x_1^{\pm}, x_2^{\pm}\right). \tag{3.117}$$

It is not difficult to show that $\Sigma(z_1, z_2)$ should satisfy the following crossing equations[30]:

$$\begin{aligned} &\Sigma(z_1, z_2)\Sigma(z_1 + \omega_2, z_2) = h\left(x_1^{\pm}, x_2^{\pm}\right),\\ &\Sigma(z_1, z_2)\Sigma(z_1, z_2 - \omega_2) = h\left(x_1^{\pm}, x_2^{\pm}\right), \end{aligned} \tag{3.118}$$

where the function $h\left(x_1^{\pm}, x_2^{\pm}\right)$ is given by

$$h\left(x_1^{\pm}, x_2^{\pm}\right) = \frac{\left(x_1^- - x_2^+\right)\left(1 - \frac{1}{x_1^- x_2^-}\right)}{\left(x_1^+ - x_2^+\right)\left(1 - \frac{1}{x_1^+ x_2^-}\right)}.$$

It is important to note that the function $h\left(x_1^{\pm}, x_2^{\pm}\right)$ obeys the following identities:

$$h\left(1/x_1^{\pm}, x_2^{\pm}\right)h\left(x_2^{\pm}, x_1^{\pm}\right) = 1, \qquad h\left(x_1^{\pm}, 1/x_2^{\pm}\right)h\left(x_2^{\pm}, x_1^{\pm}\right) = 1, \tag{3.119}$$

which are incompatible with the assumption that the dressing factor is both unitary and meromorphic function of z_i. Since unitarity is a physical requirement, the dressing factor cannot be a meromorphic function of the torus rapidity variables.

3.5. Appendix

3.5.1. Monodromies of the S-matrix. The canonical $\mathfrak{su}(2|2)$-invariant fundamental *S*-matrix is defined on a product of two rapidity tori. As such, it exhibits certain monodromy properties under shifts of rapidity variables by certain fractions of the real and imaginary periods of the torus.

By using the explicit form (3.84), one finds

$$\begin{aligned} S(z_1 + 2\omega_1, z_2) &= \Sigma_1 S(z_1, z_2)\Sigma_1 = \Sigma_2 S(z_1, z_2)\Sigma_2,\\ S(z_1 + 2\omega_2, z_2) &= \Sigma_1 S(z_1, z_2)\Sigma_1 = \Sigma_2 S(z_1, z_2)\Sigma_2. \end{aligned} \tag{3.120}$$

Hence, the *S*-matrix has the same monodromies over real and imaginary cycles and it is a periodic function on a double torus with periods $4\omega_1$ and $4\omega_2$. Here $\Sigma_1 = \Sigma \otimes \mathbb{1}$ and $\Sigma_2 = \mathbb{1} \otimes \Sigma$, where Σ is given by equation (3.81). The element Σ is in the center of the group $SU(2) \times SU(2)$. We recall that compatibility of scattering with statistics implies that

$$[S(z_1, z_2), \Sigma \otimes \Sigma] = 0. \tag{3.121}$$

Now we establish the monodromy properties with respect to shifts by half-periods. Under the shift by the real half-period we get

$$\begin{aligned} S(z_1 + \omega_1, z_2) &= (V \otimes \Sigma)S(z_1, z_2)(V^{-1} \otimes \mathbb{1}),\\ S(z_1, z_2 + \omega_1) &= (\Sigma \otimes V^{-1})S(z_1, z_2)(\mathbb{1} \otimes V) \end{aligned} \tag{3.122}$$

and, as a consequence,

$$S(z_1 + \omega_1, z_2 + \omega_1) = (\Sigma \otimes \Sigma)(V \otimes V^{-1})S(z_1, z_2)(V^{-1} \otimes V).$$

[30] The second equation in (3.118) follows from the first one by using the unitarity condition $\Sigma(z_1, z_2)\Sigma(z_2, z_1) = 1$.

Here $V = \mathrm{diag}\big(\mathrm{e}^{\frac{\mathrm{i}\pi}{4}}, \mathrm{e}^{\frac{\mathrm{i}\pi}{4}}, \mathrm{e}^{-\frac{\mathrm{i}\pi}{4}}, \mathrm{e}^{-\frac{\mathrm{i}\pi}{4}}\big)$. Thus, up to multiplication by $\Sigma \otimes \Sigma$, under the simultaneous shift of the rapidity variables by the real half-period the S-matrix undergoes a similarity transformation with $V \otimes V^{-1}$.

The shift by the imaginary half-period is the crossing transformation which has been already discussed in section 3.4. For completeness, we present it here for the S-matrix (3.84)

$$\begin{aligned} \mathscr{C}_1^{-1} S^{t_1}(z_1, z_2) \mathscr{C}_1 &= \frac{1}{f(z_1, z_2)} S^{-1}(z_1 + \omega_2, z_2), \\ \mathscr{C}_2^{-1} S^{t_2}(z_1, z_2) \mathscr{C}_2 &= \frac{1}{f(z_1, z_2)} S^{-1}(z_1, z_2 - \omega_2), \end{aligned} \tag{3.123}$$

where the function $f(z_1, z_2) \equiv f(x_1^{\pm}, x_2^{\pm})$ is defined in (3.116). Combining the last formulae and using the parity invariance of the S-matrix, we further find that

$$\mathscr{C}_1^{-1}\mathscr{C}_2^{-1} S^t(z_1, z_2) \mathscr{C}_1 \mathscr{C}_2 = S(z_1 + \omega_2, z_2 + \omega_2), \tag{3.124}$$

where $S^t(z_1, z_2) = S^{t_1,t_2}(z_1, z_2)$. Here we have used that $f(z_1, z_2) f(-z_1 - \omega_2, -z_2) = 1$. On the other hand, on can independently verify that

$$S^t(z_1, z_2) = \mathscr{C}_1\mathscr{C}_2 S(z_1, z_2) \mathscr{C}_1^{-1}\mathscr{C}_2^{-1} = \mathscr{C}_1^{-1}\mathscr{C}_2^{-1} S(z_1, z_2) \mathscr{C}_1\mathscr{C}_2. \tag{3.125}$$

These two expressions for the transposed S-matrix are compatible due to the fact that $\mathscr{C}^2 = \Sigma$. Equation (3.125) together with equation (3.124) implies that the S-matrix remains invariant under the simultaneous shift of z_1 and z_2 by ω_2

$$S(z_1 + \omega_2, z_2 + \omega_2) = S(z_1, z_2). \tag{3.126}$$

Finally, we note that the time reversal invariance and equation (3.125) lead to another commutativity property

$$[S(z_1, z_2), \mathbb{1}^g(\mathscr{C} \otimes \mathscr{C})] = 0. \tag{3.127}$$

We remark that for the S-matrix (3.84) both equations, (3.121) and (3.127), are trivially satisfied without invoking the explicit form of the coefficients a_i.

The monodromic properties of the S-matrix together with generalized physical unitarity allow one to consistently define an elliptic analog of the ZF algebra (the ZF algebra on the rapidity torus). We however will not consider it here.

3.5.2. One-loop S-matrix. Here we describe the properties of the 'one-loop' S-matrix which is obtained from the S-matrix (3.84) upon taking the limit $g \to 0$. We continue to work in the elliptic parametrization introduced in subsection 3.2.4. According to equation (3.68), in this limit Jacobi elliptic functions degenerate into the corresponding trigonometric ones and we find the following trigonometric S-matrix:

$$\begin{aligned} S(z_1, z_2) = {}& \big(E_1^1 \otimes E_1^1 + E_2^2 \otimes E_2^2 + E_1^1 \otimes E_2^2 + E_2^2 \otimes E_1^1\big) \\ &+ \frac{2\mathrm{i}}{\cot z_1 - \cot z_2 - 2\,\mathrm{i}}\big(E_1^1 \otimes E_2^2 + E_2^2 \otimes E_1^1 - E_1^2 \otimes E_2^1 - E_2^1 \otimes E_1^2\big) \\ &- \mathrm{e}^{-\mathrm{i}(z_1 - z_2)} \frac{\cot z_1 - \cot z_2 + 2\,\mathrm{i}}{\cot z_1 - \cot z_2 - 2\,\mathrm{i}}\big(E_3^3 \otimes E_3^3 + E_4^4 \otimes E_4^4 + E_3^3 \otimes E_4^4 + E_4^4 \otimes E_3^3\big) \\ &+ \mathrm{e}^{-\mathrm{i}(z_1 - z_2)} \frac{2\mathrm{i}}{\cot z_1 - \cot z_2 - 2\,\mathrm{i}}\big(E_3^3 \otimes E_4^4 + E_4^4 \otimes E_3^3 - E_4^3 \otimes E_3^4 - E_3^4 \otimes E_4^3\big) \\ &+ \mathrm{e}^{-\mathrm{i}z_1} \frac{\cot z_1 - \cot z_2}{\cot z_1 - \cot z_2 - 2\,\mathrm{i}}\big(E_1^1 \otimes E_3^3 + E_1^1 \otimes E_4^4 + E_2^2 \otimes E_3^3 + E_2^2 \otimes E_4^4\big) \\ &+ \mathrm{e}^{\mathrm{i}z_2} \frac{\cot z_1 - \cot z_2}{\cot z_1 - \cot z_2 - 2\,\mathrm{i}}\big(E_3^3 \otimes E_1^1 + E_4^4 \otimes E_1^1 + E_3^3 \otimes E_2^2 + E_4^4 \otimes E_2^2\big) \end{aligned}$$

$$- e^{-\frac{i}{2}(z_1 - z_2)} \frac{2i}{\cot z_1 - \cot z_2 - 2i} \left(E_1^3 \otimes E_3^1 + E_1^4 \otimes E_4^1 + E_2^3 \otimes E_3^2 + E_2^4 \otimes E_4^2 \right)$$
$$- e^{-\frac{i}{2}(z_1 - z_2)} \frac{2i}{\cot z_1 - \cot z_2 - 2i} \left(E_3^1 \otimes E_1^3 + E_4^1 \otimes E_1^4 + E_3^2 \otimes E_2^3 + E_4^2 \otimes E_2^4 \right). \tag{3.128}$$

The relations between the z-variable, the momentum and the rescaled rapidity $u \to gu$ transform in the limit $g \to 0$ into

$$p = 2z, \qquad u = \cot z = \cot \frac{p}{2}. \tag{3.129}$$

Surprisingly enough, this S-matrix cannot be written in the difference form, i.e. as a function of one variable being the difference of a properly introduced spectral parameter. By construction, this S-matrix satisfies the Yang–Baxter equation

$$S_{23}(z_2, z_3) S_{13}(z_1, z_3) S_{12}(z_1, z_2) = S_{12}(z_1, z_2) S_{13}(z_1, z_3) S_{23}(z_2, z_3), \tag{3.130}$$

as one can also verify by direct calculation. On the other hand, at one-loop there is another 'canonical' S-matrix which is a linear combination of the graded identity and the usual permutation

$$S_{12}^{\mathrm{can}} = \frac{u_1 - u_2}{u_1 - u_2 - 2\,i} \mathbb{1}_{12}^{g} - \frac{2i}{u_1 - u_2 - 2i} P_{12}. \tag{3.131}$$

This S-matrix satisfies the same Yang–Baxter equation (3.130).

It appears that two S-matrices, (3.128) and (3.131), are related by the following transformation:

$$S^{\mathrm{can}}(z_1, z_2) = U_2(z_1)[V_1(z_1) V_2(z_2) S_{12}(z_1, z_2) V_1^{-1}(z_1) V_2^{-1}(z_2)] U_1^{-1}(z_2),$$

where we have introduced the diagonal matrices

$$U(z) = \mathrm{diag}(1, 1, e^{iz}, e^{iz}),$$
$$V(z) = \mathrm{diag}(e^{i\frac{z}{4}}, e^{i\frac{z}{4}}, e^{-i\frac{z}{4}}, e^{-i\frac{z}{4}}).$$

The transformation by V is a 'gauge' transformation which always preserves the Yang–Baxter equation. On the other hand, transformation by U is a twist that generically transforms the usual Yang–Baxter equation into the twisted one and vice versa. Note also that the twist U does not belong to the symmetry group $\mathrm{SU}(2) \times \mathrm{SU}(2)$ of the 'all-loop' S-matrix.

To understand why at one loop the Yang–Baxter equation is preserved under the twisting, we first write the Yang–Baxter equation for S^{can} by using[31] equation (3.131)

$$U_3(z_2) S_{23} U_2^{-1}(z_3) U_3(z_1) S_{13} U_1^{-1}(z_3) U_2(z_1) S_{12} U_1^{-1}(z_2)$$
$$= U_2(z_1) S_{12} U_1^{-1}(z_2) U_3(z_1) S_{13} U_1^{-1}(z_3) U_3(z_2) S_{23} U_2^{-1}(z_3), \tag{3.132}$$

which can be reshuffled as follows:

$$U_3(z_2) S_{23} U_2(z_1) U_3(z_1) S_{13} U_1^{-1}(z_3) U_2^{-1}(z_3) S_{12} U_1(z_2)$$
$$= U_2(z_1) U_3(z_1) S_{12} U_1^{-1}(z_2) S_{13} U_3(z_2) S_{23} U_1^{-1}(z_3) U_2^{-1}(z_3). \tag{3.133}$$

It is clear now that we will get the usual Yang–Baxter equation for S provided it obeys the following relation:

$$[S, U \otimes U] = 0, \tag{3.134}$$

[31] The gauge transformation by the matrix V decouples from the Yang–Baxter equation.

where U is an *arbitrary diagonal matrix*. One can easily verify that both S-matrices, (3.128) and (3.131), do indeed satisfy this relation. At higher orders in g the relation (3.134) does not hold anymore. The corresponding all-loop S-matrix (3.84) satisfies only a weaker condition

$$[S, G\otimes G] = 0, \qquad G \in \mathrm{SU}(2)\times\mathrm{SU}(2), \tag{3.135}$$

which is nothing else but the invariance condition. As a consequence, the Yang–Baxter equation is preserved by the twist only at the one-loop order.

3.5.3. Hopf algebra interpretation. In section 3.3 we have determined the commutation relations of the $\mathfrak{su}(2|2)$ symmetry algebra generators with the ZF operators. This allowed us to define the action of this symmetry algebra in the multi-particle states constructed by successive application of creation operators. An alternative way to define this action is to use the concept of a Hopf algebra.

Let $\mathcal{A}$ be a vector space over complex numbers. Consider the following two maps

$$\Delta : \mathcal{A} \to \mathcal{A}\otimes\mathcal{A}, \qquad \epsilon : \mathcal{A} \to \text{ complex numbers.} \tag{3.136}$$

If these maps satisfy the relations

$$(\mathrm{id}\otimes\Delta)\circ\Delta = (\Delta\otimes\mathrm{id})\circ\Delta, \qquad (\mathrm{id}\otimes\epsilon)\circ\Delta = \mathrm{id} = (\epsilon\otimes\mathrm{id})\circ\Delta, \tag{3.137}$$

then $\mathcal{A}$ is called a coalgebra. Accordingly, the map Δ is called the coproduct (or comultiplication) of $\mathcal{A}$ and ϵ is the counit of $\mathcal{A}$. A bialgebra $\mathcal{A}$ is both a unital associative algebra and a coalgebra such that Δ and ϵ are algebra homomorphisms, and multiplication μ and identity $\mathbb{1}$ are coalgebra homomorphisms. The fact that Δ and ϵ are algebra homomorphisms is expressed as

$$\Delta(ab) = \Delta(a)\Delta(b), \qquad \epsilon(ab) = \epsilon(a)\epsilon(b), \qquad a, b \in \mathcal{A},$$

Finally, a Hopf algebra is a bialgebra equipped with a bijective map $S : \mathcal{A} \to \mathcal{A}$, called antipode, obeying the following relations:

$$\mu(S\otimes\mathrm{id})\circ\Delta = \mathbb{1}\circ\epsilon = \mu(\mathrm{id}\otimes S)\circ\Delta.$$

Let now $\mathcal{A}$ be a unitary graded associative algebra generated by even rotation generators $\mathbb{L}_a{}^b$, $\mathbb{R}_\alpha{}^\beta$, the odd supersymmetry generators $\mathbb{Q}_\alpha{}^a$, $\mathbb{Q}^{\dagger}{}_a^{\alpha}$ and two central elements $\mathbb{H}$ and $\mathbb{P}$ subject to equations (3.43). The central charges $\mathbb{C}$ and $\mathbb{C}^\dagger$ are expressed via $\mathbb{P}$ by means of equations (1.115).

In what follows we make use of the graded tensor product, that is for any algebra elements a, b, c, d

$$(a\hat{\otimes}b)(c\hat{\otimes}d) = (-1)^{\epsilon_b\epsilon_c}(ac\hat{\otimes}bd),$$

where $\epsilon_a = 0$ if a is even and $\epsilon_a = -1$ if a is odd.

Now we are ready to supply $\mathcal{A}$ with the structure of a Hopf algebra. We define the following coproduct:

$$\begin{aligned}
&\Delta(\mathbb{J}) = \mathbb{J}\hat{\otimes}\mathbb{1} + \mathbb{1}\hat{\otimes}\mathbb{J} \qquad \text{for any even generator,}\\
&\Delta(\mathbb{Q}_\alpha{}^a) = \mathbb{Q}_\alpha{}^a\hat{\otimes}\mathbb{1} + \mathrm{e}^{\frac{\mathrm{i}}{2}\mathbb{P}}\hat{\otimes}\mathbb{Q}_\alpha{}^a,\\
&\Delta(\mathbb{Q}^{\dagger}{}_a^{\alpha}) = \mathbb{Q}^{\dagger}{}_a^{\alpha}\hat{\otimes}\mathbb{1} + \mathrm{e}^{-\frac{\mathrm{i}}{2}\mathbb{P}}\hat{\otimes}\mathbb{Q}^{\dagger}{}_a^{\alpha},
\end{aligned} \tag{3.138}$$

the counit

$$\epsilon(\mathbb{1}) = 1, \qquad \epsilon(\mathbb{J}) = \epsilon\big(\mathbb{Q}_\alpha{}^a\big) = \epsilon\big(\mathbb{Q}^{\dagger}{}_a^{\alpha}\big) = 0 \tag{3.139}$$

and the antipode

$$S(\mathbb{J}) = -\mathbb{J}, \qquad S\big(\mathbb{Q}_\alpha{}^a\big) = -\mathrm{e}^{-\frac{\mathrm{i}}{2}\mathbb{P}}\mathbb{Q}_\alpha{}^a, \qquad S\big(\mathbb{Q}^{\dagger}{}_a^{\alpha}\big) = -\mathrm{e}^{\frac{\mathrm{i}}{2}\mathbb{P}}\mathbb{Q}^{\dagger}{}_a^{\alpha} \tag{3.140}$$

and $S(\mathbb{1}) = 1$. The reader can easily verify that with these definitions all Hopf algebra axioms are satisfied. For instance, we compute[32]

$$\Delta(\mathbb{C}) = \frac{ig}{2}(e^{\mathbb{P}\otimes\mathbb{1}+\mathbb{1}\otimes\mathbb{P}} - \mathbb{1}\otimes\mathbb{1}) = \frac{ig}{2}(e^{\mathbb{P}}\otimes e^{\mathbb{P}} - \mathbb{1}\otimes\mathbb{1}) = \mathbb{C}\otimes\mathbb{1} + e^{i\mathbb{P}}\otimes\mathbb{C}.$$

On the other hand,

$$\left\{\Delta\mathbb{Q}_\alpha{}^a, \Delta\mathbb{Q}_\beta{}^b\right\} = \left\{\mathbb{Q}_\alpha{}^a\hat{\otimes}\mathbb{1} + e^{\frac{i}{2}\mathbb{P}}\hat{\otimes}\mathbb{Q}_\alpha{}^a, \mathbb{Q}_\beta{}^b\hat{\otimes}\mathbb{1} + e^{\frac{i}{2}\mathbb{P}}\hat{\otimes}\mathbb{Q}_\beta{}^b\right\} =$$
$$\left\{\mathbb{Q}_\alpha{}^a, \mathbb{Q}_\beta{}^b\right\}\otimes\mathbb{1} + e^{i\mathbb{P}}\otimes\left\{\mathbb{Q}_\alpha{}^a, \mathbb{Q}_\beta{}^b\right\} = \epsilon_{\alpha\beta}\epsilon^{ab}(\mathbb{C}\otimes\mathbb{1} + e^{i\mathbb{P}}\otimes\mathbb{C}) = \epsilon_{\alpha\beta}\epsilon^{ab}\Delta\mathbb{C},$$

i.e. Δ is indeed an algebra homomorphism.

Let us show that the coproduct agrees with the form of the two-particle structure constants appearing in (3.80). Let $\mathscr{V}$ be a vector space of the fundamental representation of $\mathcal{A}$. This space has a natural grading; the corresponding grading matrix is given by Σ. The action of, say, supersymmetry generators $\mathbb{Q}_\alpha{}^a$ on the tensor product $\mathscr{V}\otimes\mathscr{V}$ is given by application of the coproduct (3.138)

$$\begin{aligned}\Delta\big(\mathbb{Q}_\alpha{}^a\big)\cdot v\otimes u &= \big(\mathbb{Q}_\alpha{}^a\hat{\otimes}\mathbb{1} + e^{\frac{i}{2}\mathbb{P}}\hat{\otimes}\mathbb{Q}_\alpha{}^a\big)\cdot v\otimes u\\ &= \mathbb{Q}_\alpha{}^a\cdot v\otimes u + \Sigma\, e^{\frac{i}{2}\mathbb{P}}\cdot v\otimes\mathbb{Q}_\alpha{}^a\cdot u,\end{aligned} \qquad (3.141)$$

where $v\otimes u$ is an element of $\mathscr{V}\otimes\mathscr{V}$. Now one can recognize that the two-particle representation coincides with the one appearing on the left-hand side of (3.80).

The action of the Hopf algebra operations on the algebra generators depends on the chosen bases. Recall that $\mathcal{A}$ admits an automorphism

$$\mathbb{Q}\to e^{i\xi}\mathbb{Q}, \qquad \mathbb{C}\to e^{2i\xi}\mathbb{C},$$

where ξ might be a non-trivial function of the central charges. For the choice $\xi = -\frac{1}{4}\mathbb{P}$ the central charges $\mathbb{C}$ and $\mathbb{C}^\dagger$ take the form (2.134), and they become real and coincide. The action of the coproduct on the redefined supercharges takes the most symmetric form

$$\begin{aligned}\Delta(\mathbb{Q}_\alpha{}^a) &= \mathbb{Q}_\alpha{}^a\hat{\otimes}\, e^{-\frac{i}{4}\mathbb{P}} + e^{\frac{i}{4}\mathbb{P}}\hat{\otimes}\mathbb{Q}_\alpha{}^a,\\ \Delta(\mathbb{Q}^{\dagger\alpha}_a) &= \mathbb{Q}^{\dagger\alpha}_a\hat{\otimes}\, e^{\frac{i}{4}\mathbb{P}} + e^{-\frac{i}{4}\mathbb{P}}\hat{\otimes}\mathbb{Q}^{\dagger\alpha}_a.\end{aligned}$$

In the new basis the antipod becomes trivial for any algebra element

$$S(\mathbb{J}) = -\mathbb{J}, \qquad S\big(\mathbb{Q}_\alpha{}^a\big) = -\mathbb{Q}_\alpha{}^a, \qquad S\big(\mathbb{Q}^{\dagger\alpha}_a\big) = -\mathbb{Q}^{\dagger\alpha}_a. \qquad (3.142)$$

The only drawback of this algebra basis is that with $\mathbb{C}$ real a basis of the corresponding fundamental representation cannot depend meromorphically on the torus variable z.

Our final comment concerns permutation relations (3.87). We observe that they can be cast in the usual (anti)-commutator form by redefining the supersymmetry generators $\mathbb{Q}_\alpha{}^a$ and $\mathbb{Q}^{\dagger\alpha}_a$ in the following way:

$$\mathbb{Q}_a{}^\alpha \to \mathbb{Q}_a{}^\alpha e^{i\mathbb{P}/2}, \qquad \mathbb{Q}^{\dagger\alpha}_a \to \mathbb{Q}^{\dagger\alpha}_a e^{-i\mathbb{P}/2}. \qquad (3.143)$$

Relations (3.87) for the redefined supersymmetry charges take the form of the (anti)-commutators

$$\begin{aligned}\mathbb{Q}_\alpha{}^a\mathbb{A}^\dagger(p) - \mathbb{A}^\dagger(p)\Sigma\mathbb{Q}_\alpha{}^a &= \mathbb{A}^\dagger(p)Q_\alpha{}^a(p)\, e^{-i\mathbb{P}/2},\\ \mathbb{Q}^{\dagger\alpha}_a\mathbb{A}^\dagger(p) - \mathbb{A}^\dagger(p)\Sigma\mathbb{Q}^{\dagger\alpha}_a &= \mathbb{A}^\dagger(p)Q^{\dagger\alpha}_a(p)\, e^{i\mathbb{P}/2}.\end{aligned} \qquad (3.144)$$

The only difference with the standard relations is the appearance of the operator $e^{\pm i\mathbb{P}/2}$ in the right-hand side of equations (3.144). As in our discussion above, redefinition (3.143) changes the momentum dependence of the central charge $\mathbb{C}$

$$\mathbb{C}\to\frac{ig}{2}(e^{i\mathbb{P}} - 1)\, e^{-i\mathbb{P}} = \frac{ig}{2}(1 - e^{-i\mathbb{P}}), \qquad (3.145)$$

[32] Since all elements here are even we can use the usual tensor product.

and, therefore, the boundary conditions for the light-cone coordinate x_-. Obviously, it does not change the form of the S-matrix if one keeps track of the additional phases because the redefined supercharges also commute with the S-matrix.

3.6. Bibliographic remarks

The Factorized Scattering Theory has been developed in [28]. For important applications the reader may consult [105, 106]. The ZF algebra has been introduced in [28, 107]. Its various properties and representation theory have been extensively discussed in the literature, see, e.g., [108, 109]. Our exposition of the Factorized Scattering Theory and its application to the string sigma model follows closely [26, 27].

The exact dispersion relation (3.56) has been conjectured in [17]. In this work the local conserved charges (3.96) has been introduced as the 'higher loop' generalization of conserved charges of the Heisenberg model.

The $\mathfrak{psu}(2|2)$-invariant S-matrix has been obtained in [16] by exploiting the corresponding invariance condition. This condition severely constraints its matrix structure but does not fix it uniquely. In general, the S-matrix depends on a few parameters [38], which reflects the freedom of choice of a two-particle basis, and, as a result, it satisfies a twisted version of the Yang–Baxter equation. In a physical theory the S-matrix must be unique (up to unitary transformations). In two-dimensional integrable models it must satisfy the condition of factorized scattering, i.e. the Yang–Baxter equation. This requirement partially fixes the two-particle basis and the corresponding S-matrix [26] leaving the possibility of performing momentum-dependent transformations of one-particle states. An additional requirement of generalized physical unitarity, or, equivalently, of physical unitarity of the S-matrix of the mirror model, leads to a unique matrix expression [27] up to constant transformations of the one-particle basis. Then, the only undetermined piece of the S-matrix is an overall normalization (the scalar factor). The graded S-matrix obtained in section 3 is the inverse of the graded version of S found in [26, 27]. This is done to get an agreement with the perturbative S-matrix of section 2, the latter was computed by using the standard field-theoretic prescriptions.

The idea that the overall scalar factor can be constrained by requiring the world-sheet scattering matrix to satisfy an analog of crossing symmetry has been put forward in [29], where also a functional equation for this factor implied by crossing symmetry has been derived. In fact, in relativistic integrable models compatibility of scattering with crossing symmetry is a standard requirement [28, 110, 111]. In this respect, a peculiarity of the string sigma model lies in the absence of the two-dimensional Lorentz invariance on the world-sheet. In the last section we exhibited three different faces of crossing symmetry: crossing symmetry as an additional invariance condition for the ZF algebra [26], crossing symmetry as a requirement of trivial scattering of the singlet state [38] (see also [112]) and, finally, crossing symmetry as the particle-to-antiparticle transform [29].

The representation theory of the centrally extended $\mathfrak{su}(2|2)$ algebra has been studied in [38], where, in particular, conditions leading to the multiplet shortening have been determined and the outer automorphism group SL(2) has been identified. The rapidity torus has been introduced in [29], although our uniformization (the same uniformization has been also used in [38]) for the dispersion relation in terms of elliptic functions is different from that in [29]. Table 1 representing the transformation properties of $x^{\pm}(z)$ under shifts of z by some fractions of the periods is taken from [38].

The most non-trivial part of the overall scalar factor of the world-sheet S-matrix is the dressing phase. Its functional form in terms of local conserved charges of the model was conjectured in [18] by discretizing the finite-gap solutions [22] of the classical string sigma model. The most general functional form of the dressing phase compatible with integrability [113] is given by equation (3.95).

Further progress in determination of the dressing phase relied on comparison of the energies of spinning strings at the classical and the one-loop level [24, 25, 114–120] with those obtained by solving the asymptotic Bethe ansatz equations [121–129]. The general method for determining the one-loop correction to the dressing phase has been developed in [132] and used to obtain the one-loop correction to the coefficient $c_{2,3}$ in equation (3.95). This approached has been further applied to completely determine the dressing phase at one loop [30, 133]. The same results were later derived by using the algebraic curve techniques [130, 131].

Two known orders in the strong coupling expansion of the dressing phase [18, 30, 133] were shown to solve the functional equation implied by crossing symmetry [100]. Formula (3.100) that encodes an all-order asymptotic solution for the dressing phase was obtained in [31] by exploiting its functional form (3.95) together with the crossing equation. Opposite to the strong coupling expansion, gauge theory perturbative expansion of the dressing factor is in powers of g and it has a finite radius of convergence. A proposal (3.103) leading to the exact dressing factor has been put forward in [34] and it passed several very non-trivial tests [32–36, 134]. A check that this exact dressing phase obeys the crossing equation for finite values of g, i.e. not in the asymptotic sense, is currently lacking, however.

As was discussed at the beginning of section 3, quantum integrability is a plausible but yet unproven property of the string sigma model. To reveal it, one has to demonstrate the absence of particle production and factorization of multi-particle scattering. This important question has been investigated in [135, 136], where factorization has been shown to hold at leading orders in the strong coupling expansion.

The monodromy properties of the $\mathfrak{su}(2|2)$-invariant S-matrix have been established in [27], where an elliptic analog of the ZF algebra has been introduced. The one-loop limit of the S-matrix and its relation to the canonical S-matrix built out of the graded identity and the permutation has been also analyzed there. The Hopf algebra structure discussed in the appendix and in [26] seems to be equivalent (up to a twist and some redefinitions of the supersymmetry generators and the central elements) to the one studied in [137, 138].

Acknowledgments

We thank F Alday, N Beisert, J Plefka, R Roiban, M Staudacher, A Tseytlin and M Zamaklar for enjoyable collaborations. We are also grateful to N Dorey, P Dorey, D Hofman, R Janik, V Kazakov, C Kristjansen, M de Leeuw, J Maldacena, J Minahan and K Zarembo for many valuable discussions. We also thank M de Leeuw, E Quinn and A Torrielli for the careful reading of the manuscript and M de Leeuw for helping with figures. The work of GA was supported in part by the RFBI grant 08-01-00281-a, by the grant NSh-672.2006.1, by NWO grant 047017015 and by the INTAS contract 03-51-6346. The work of SF was supported in part by the Science Foundation Ireland under grant no 07/RFP/PHYF104.

References

[1] Maldacena J M 1998 The large N limit of superconformal field theories and supergravity *Adv. Theor. Math. Phys.* **2** 231

Maldacena J M 1999 The large N limit of superconformal field theories and supergravity *Int. J. Theor. Phys.* **38** 1113 (arXiv:hep-th/9711200)

[2] Gubser S S, Klebanov I R and Polyakov A M 1998 Gauge theory correlators from non-critical string theory *Phys. Lett.* B **428** 105 (arXiv: hep-th/9802109)

[3] Witten E 1998 Anti-de Sitter space and holography *Adv. Theor. Math. Phys.* **2** 253 (arXiv:hep-th/9802150)

[4] Berenstein D, Maldacena J M and Nastase H 2002 Strings in flat space and pp waves from $N = 4$ super Yang Mills *J. High Energy Phys.* JHEP04(2002)013 (arXiv:hep-th/0202021)

[5] Minahan J A and Zarembo K 2003 The Bethe-ansatz for $N = 4$ super Yang–Mills *J. High Energy Phys.* JHEP03(2003)013 (arXiv:hep-th/0212208)

[6] Beisert N, Kristjansen C and Staudacher M 2003 The dilatation operator of $N = 4$ super Yang–Mills theory *Nucl. Phys.* B **664** 131 (arXiv:hep-th/0303060)

[7] Beisert N and Staudacher M 2003 The $N = 4$ SYM integrable super spin chain *Nucl. Phys.* B **670** 439 (arXiv:hep-th/0307042)

[8] Bena I, Polchinski J and Roiban R 2004 Hidden symmetries of the $AdS_5 \times S^5$ superstring *Phys. Rev.* D **69** 046002 (arXiv:hep-th/0305116)

[9] Green M B, Schwarz J H and Witten E 1987 *Superstring Theory: Introduction (Cambridge Monographs On Mathematical Physics* vol 1*)* (Cambridge: Cambridge University Press) p 469

[10] Belitsky A V, Braun V M, Gorsky A S and Korchemsky G P 2004 Integrability in QCD and beyond *Int. J. Mod. Phys.* A **19** 4715 (arXiv:hep-th/0407232)

[11] Beisert N 2005 The dilatation operator of $N = 4$ super Yang–Mills theory and integrability *Phys. Rep.* **405** 1 (arXiv:hep-th/0407277)

[12] Metsaev R R and Tseytlin A A 1998 Type IIB superstring action in $AdS_5 \times S^5$ background *Nucl. Phys.* B **533** 109 (arXiv:hep-th/9805028)

[13] Arutyunov G and Frolov S 2005 Integrable Hamiltonian for classical strings on $AdS_5 \times S^5$ *J. High Energy Phys.* JHEP02(2005)059 (arXiv:hep-th/0411089)

[14] Frolov S, Plefka J and Zamaklar M 2006 The $AdS_5 \times S^5$ superstring in light-cone gauge and its Bethe equations *J. Phys. A: Math. Gen.* **39** 13037 (arXiv:hep-th/0603008)

[15] Arutyunov G, Frolov S, Plefka J and Zamaklar M 2007 The off-shell symmetry algebra of the light-cone $AdS_5 \times S^5$ superstring *J. Phys. A: Math. Theor.* **40** 3583 (arXiv:hep-th/0609157)

[16] Beisert N 2008 The su(2|2) dynamic S-matrix *Adv. Theor. Math. Phys.* **12** 945 (arXiv:hep-th/0511082)

[17] Beisert N, Dippel V and Staudacher M 2004 A novel long range spin chain and planar $N = 4$ super Yang–Mills *J. High Energy Phys.* JHEP07(2004)075 (arXiv:hep-th/0405001)

[18] Arutyunov G, Frolov S and Staudacher M 2004 Bethe ansatz for quantum strings *J. High Energy Phys.* JHEP10(2004)016 (arXiv:hep-th/0406256)

[19] Staudacher M 2005 The factorized S-matrix of CFT/AdS *J. High Energy Phys.* JHEP05(2005)054 (arXiv:hep-th/0412188)

[20] Sutherland B 1978 A brief hystory of the quantum soliton with new results on the quantization of the Toda lattice *Rocky Mt. J. Math.* **8** 431

[21] Beisert N and Staudacher M 2005 Long-range PSU(2,2|4) Bethe ansaetze for gauge theory and strings *Nucl. Phys.* B **727** 1 (arXiv:hep-th/0504190)

[22] Kazakov V A, Marshakov A, Minahan J A and Zarembo K 2004 Classical/quantum integrability in AdS/CFT *J. High Energy Phys.* JHEP05(2004)024 (arXiv:hep-th/0402207)

[23] Gubser S S, Klebanov I R and Polyakov A M 2002 A semi-classical limit of the gauge/string correspondence *Nucl. Phys.* B **636** 99 (arXiv:hep-th/0204051)

[24] Frolov S and Tseytlin A A 2002 Semiclassical quantization of rotating superstring in $AdS_5 \times S^5$ *J. High Energy Phys.* JHEP06(2002)007 (arXiv:hep-th/0204226)

[25] Frolov S and Tseytlin A A 2003 Multi-spin string solutions in $AdS_5 \times S^5$ *Nucl. Phys.* B **668** 77 (arXiv:hep-th/0304255)

[26] Arutyunov G, Frolov S and Zamaklar M 2007 The Zamolodchikov–Faddeev algebra for $AdS_5 \times S^5$ superstring *J. High Energy Phys.* JHEP04(2007)002 (arXiv:hep-th/0612229)

[27] Arutyunov G and Frolov S 2007 On string S-matrix, bound states and TBA *J. High Energy Phys.* JHEP12(2007)024 (arXiv:0710.1568[hep-th])

[28] Zamolodchikov A B and Zamolodchikov A B 1979 Factorized S-matrices in two dimensions as the exact solutions of certain relativistic quantum field models *Ann. Phys.* **120** 253

[29] Janik R A 2006 The $AdS_5 \times S^5$ superstring worldsheet S-matrix and crossing symmetry *Phys. Rev.* D **73** 086006 (arXiv:hep-th/0603038)

[30] Hernandez R and Lopez E 2006 Quantum corrections to the string Bethe ansatz *J. High Energy Phys.* JHEP07(2006)004 (arXiv:hep-th/0603204)

[31] Beisert N, Hernandez R and Lopez E 2006 A crossing-symmetric phase for $AdS_5 \times S^5$ strings *J. High Energy Phys.* JHEP11(2006)070 (arXiv:hep-th/0609044)

[32] Roiban R, Tirziu A and Tseytlin A A 2007 Two-loop world-sheet corrections in $AdS_5 \times S^5$ superstring *J. High Energy Phys.* JHEP07(2007)056 (arXiv:0704.3638[hep-th])
[33] Klose T, McLoughlin T, Minahan J A and Zarembo K 2007 World-sheet scattering in $AdS_5 \times S^5$ at two loops *J. High Energy Phys.* JHEP08(2007)051 (arXiv:0704.3891[hep-th])
[34] Beisert N, Eden B and Staudacher M 2007 Transcendentality and crossing *J. Stat. Mech.* **0701** P021 (arXiv:hep-th/0610251)
[35] Bern Z, Czakon M, Dixon L J, Kosower D A and Smirnov V A 2007 The four-loop planar amplitude and cusp anomalous dimension in maximally supersymmetric Yang–Mills theory *Phys. Rev.* D **75** 085010 (arXiv:hep-th/0610248)
[36] Beisert N, McLoughlin T and Roiban R 2007 The four-loop dressing phase of $N = 4$ SYM *Phys. Rev.* D **76** 046002 (arXiv:0705.0321[hep-th])
[37] Dorey N 2006 Magnon bound states and the AdS/CFT correspondence *J. Phys. A: Math. Gen.* **39** 13119 (arXiv:hep-th/0604175)
[38] Beisert N 2007 The analytic Bethe ansatz for a chain with centrally extended su(2|2) symmetry *J. Stat. Mech.* **0701** P017 (arXiv:nlin/0610017)
[39] Chen H Y, Dorey N and Okamura K 2006 On the scattering of magnon boundstates *J. High Energy Phys.* JHEP11(2006)035 (arXiv:hep-th/0608047)
[40] Roiban R 2007 Magnon bound-state scattering in gauge and string theory *J. High Energy Phys.* JHEP04(2007)048 (arXiv:hep-th/0608049)
[41] Arutyunov G and Frolov S 2008 The S-matrix of string bound states *Nucl. Phys.* B **804** 90 (arXiv:0803.4323[hep-th])
[42] Martins M J and Melo C S 2007 The Bethe ansatz approach for factorizable centrally extended S-matrices *Nucl. Phys.* B **785** 246 (arXiv:hep-th/0703086)
[43] Lüscher M 1986 Volume dependence of the energy spectrum in massive quantum field theories: 1. Stable particle states *Commun. Math. Phys.* **104** 177
[44] Lüscher M 1986 Volume dependence of the energy spectrum in massive quantum field theories: 2. Scattering states *Commun. Math. Phys.* **105** 153
[45] Janik R A and Lukowski T 2007 Wrapping interactions at strong coupling—the giant magnon *Phys. Rev.* D **76** 126008 (arXiv:0708.2208[hep-th])
[46] Bajnok Z and Janik R A 2009 Four-loop perturbative Konishi from strings and finite size effects for multiparticle states *Nucl. Phys.* B **807** 625 (arXiv:0807.0399[hep-th])
[47] Fiamberti F, Santambrogio A, Sieg C and Zanon D 2008 Wrapping at four loops in $N = 4$ SYM *Phys. Lett.* B **666** 100 (arXiv:0712.3522[hep-th])
[48] Zamolodchikov A B 1990 Thermodynamic Bethe Ansatz in relativistic models. Scaling three state Potts and Lee–Yang models *Nucl. Phys.* B **342** 695
[49] Arutyunov G and Frolov S 2009 String hypothesis for the $AdS_5 \times S^5$ mirror *J. High Energy Phys.* JHEP03(2009)152 (arXiv:0901.1417[hep-th])
[50] Zamolodchikov A B 1991 TBA equations for integrable perturbed SU(2)-k x SU(2)-l/SU(2)-k + l coset models *Nucl. Phys.* B **366** 122
[51] Kuniba A, Nakanishi T and Suzuki J 1994 Functional relations in solvable lattice models: 1. Functional relations and representation theory *Int. J. Mod. Phys.* A **9** 5215 (arXiv:hep-th/9309137)
[52] Gromov N, Kazakov V and Vieira P 2009 Integrability for the full spectrum of planar AdS/CFT arXiv:0901.3753 [hep-th]
[53] Green M B and Schwarz J H 1984 Covariant description of superstrings *Phys. Lett.* B **136** 367
[54] Brink L and Schwarz J H 1981 Quantum superspace *Phys. Lett.* B **100** 310
[55] de Azcarraga J A and Lukierski J 1982 Supersymmetric particles with internal symmetries and central charges *Phys. Lett.* B **113** 170
[56] Siegel W 1983 Hidden local supersymmetry in the supersymmetric particle action *Phys. Lett.* B **128** 397
[57] Henneaux M and Mezincescu L 1985 A sigma model interpretation of Green–Schwarz covariant superstring action *Phys. Lett.* B **152** 340
[58] Grisaru M T, Howe P S, Mezincescu L, Nilsson B and Townsend P K 1985 $N = 2$ superstrings in a supergravity background *Phys. Lett.* B **162** 116
[59] Witten E 1986 Twistor—like transform in ten-dimensions *Nucl. Phys.* B **266** 245
[60] Kallosh R and Rahmfeld J 1998 The GS string action on $AdS_5 \times S^5$ *Phys. Lett.* B **443** 143 (arXiv:hep-th/9808038)
[61] Bershadsky M, Zhukov S and Vaintrob A 1999 PSL(n|n) sigma model as a conformal field theory *Nucl. Phys.* B **559** 205 (arXiv:hep-th/9902180)

[62] Roiban R and Siegel W 2000 Superstrings on $AdS_5 \times S^5$ supertwistor space *J. High Energy Phys.* JHEP11(2000)024 (arXiv:hep-th/0010104)
[63] Berkovits N, Bershadsky M, Hauer T, Zhukov S and Zwiebach B 2000 Superstring theory on $AdS_2 \times S^2$ as a coset supermanifold *Nucl. Phys.* B **567** 61 (arXiv:hep-th/9907200)
[64] Alday L F, Arutyunov G and Tseytlin A A 2005 On integrability of classical superstrings in $AdS_5 \times S^5$ *J. High Energy Phys.* JHEP07(2005)002 (arXiv:hep-th/0502240)
[65] Kac V G 1977 Lie superalgebras *Adv. Math.* **26** 8
[66] Bars I 1983 Supergroups and their representations *Lectures Appl. Math.* **21** 17
[67] Frappat L, Sorba P and Sciarrino A 1996 Dictionary on Lie superalgebras arXiv:hep-th/9607161
[68] Serganova V V 1985 Automorphisms of simple Lie superalgebras *Math. USSR Izv.* **24** 539–51
[69] McArthur I N 2000 Kappa-symmetry of Green–Schwarz actions in coset superspaces *Nucl. Phys.* B **573** 811 (arXiv:hep-th/9908045)
[70] Faddeev L D and Takhtajan L A 1987 *Hamiltonian Methods in the Theory of Solitons Springer series in soviet mathematics)* (Berlin: Springer) 592 p
[71] Babelon O, Bernard D and Talon M 2003 *Introduction to Classical Integrable Systems* (Cambridge: Cambridge University Press)
[72] Kazakov V A and Zarembo K 2004 Classical/quantum integrability in non-compact sector of AdS/CFT *J. High Energy Phys.* JHEP10(2004)060 (arXiv:hep-th/0410105)
[73] Beisert N, Kazakov V A and Sakai K 2006 Algebraic curve for the SO(6) sector of AdS/CFT *Commun. Math. Phys.* **263** 611 (arXiv:hep-th/0410253)
[74] Schafer-Nameki S 2005 The algebraic curve of 1-loop planar $N = 4$ SYM *Nucl. Phys.* B **714** 3 (arXiv:hep-th/0412254)
[75] Das A K, Maharana J, Melikyan A and Sato M 2004 The algebra of transition matrices for the $AdS_5 \times S^5$ superstring *J. High Energy Phys.* JHEP12(2004)055 (arXiv:hep-th/0411200)
[76] Beisert N, Kazakov V A, Sakai K and Zarembo K 2006 The algebraic curve of classical superstrings on $AdS_5 \times S^5$ *Commun. Math. Phys.* **263** 659 (arXiv:hep-th/0502226)
[77] Dorey N and Vicedo B 2007 A symplectic structure for string theory on integrable backgrounds *J. High Energy Phys.* JHEP03(2007)045 (arXiv:hep-th/0606287)
[78] Magro M 2009 The classical exchange algebra of $AdS_5 \times S^5$ *J. High Energy Phys.* JHEP01(2009)021 (arXiv:0810.4136[hep-th])
[79] Polyakov A M 2004 Conformal fixed points of unidentified gauge theories *Mod. Phys. Lett.* A **19** 1649 (arXiv:hep-th/0405106)
[80] Alday L F, Arutyunov G and Frolov S 2006 New integrable system of 2dim fermions from strings on $AdS_5 \times S^5$ *J. High Energy Phys.* JHEP01(2006)078 (arXiv:hep-th/0508140)
[81] Klose T, McLoughlin T, Roiban R and Zarembo K 2007 Worldsheet scattering in $AdS_5 \times S^5$ *J. High Energy Phys.* JHEP03(2007)094 (arXiv:hep-th/0611169)
[82] Alday L F, Arutyunov G and Frolov S 2006 Green–Schwarz strings in TsT-transformed backgrounds *J. High Energy Phys.* JHEP06(2006)018 (arXiv:hep-th/0512253)
[83] Goddard P, Goldstone J, Rebbi C and Thorn C B 1973 Quantum dynamics of a massless relativistic string *Nucl. Phys.* B **56** 109
[84] Metsaev R R, Thorn C B and Tseytlin A A 2001 Light-cone superstring in AdS space-time‘ *Nucl. Phys.* B **596** 151 (arXiv:hep-th/0009171)
[85] Arutyunov G and Frolov S 2006 Uniform light-cone gauge for strings in AdS: solving su(1|1) sector *J. High Energy Phys.* JHEP01(2006)055 (arXiv:hep-th/0510208)
[86] Arutyunov G, Frolov S and Zamaklar M 2007 Finite-size effects from giant magnons *Nucl. Phys.* B **778** 1 (arXiv:hep-th/0606126)
[87] Kruczenski M, Ryzhov A V and Tseytlin A A 2004 Large spin limit of $AdS_5 \times S^5$ string theory and low energy expansion of ferromagnetic spin chains *Nucl. Phys.* B **692** 3 (arXiv:hep-th/0403120)
[88] Kruczenski M and Tseytlin A A 2004 Semiclassical relativistic strings in S^5 and long coherent operators in $N = 4$ SYM theory *J. High Energy Phys.* JHEP09(2004)038 (arXiv:hep-th/0406189)
[89] Blau M, Figueroa-O'Farrill J M, Hull C and Papadopoulos G 2002 A new maximally supersymmetric background of IIB superstring theory *J. High Energy Phys.* JHEP01(2002)047 (arXiv:hep-th/0110242)
[90] Blau M, Figueroa-O'Farrill J M, Hull C and Papadopoulos G 2002 Penrose limits and maximal supersymmetry *Class. Quantum Grav.* **19** L87 (arXiv:hep-th/0201081)
[91] Metsaev R R 2002 Type IIB Green–Schwarz superstring in plane wave Ramond–Ramond background *Nucl. Phys.* B **625** 70 (arXiv:hep-th/0112044)
[92] Metsaev R R and Tseytlin A A 2002 Exactly solvable model of superstring in plane wave Ramond–Ramond background *Phys. Rev.* D **65** 126004 (arXiv:hep-th/0202109)

[93] Parnachev A and Ryzhov A V 2002 Strings in the near plane wave background and AdS/CFT *J. High Energy Phys.* JHEP10(2002)066 (arXiv:hep-th/0208010)

[94] Callan C G, Lee H K, McLoughlin T, Schwarz J H, Swanson I and Wu X 2003 Quantizing string theory in $AdS_5 \times S^5$: beyond the pp-wave *Nucl. Phys.* B **673** 3 (arXiv:hep-th/0307032)

[95] Callan C G, McLoughlin T and Swanson I 2004 Holography beyond the Penrose limit *Nucl. Phys.* B **694** 115 (arXiv:hep-th/0404007)

[96] Callan C G, McLoughlin T and Swanson I 2004 Higher impurity AdS/CFT correspondence in the near-bMN limit *Nucl. Phys.* B **700** 271 (arXiv:hep-th/0405153)

[97] McLoughlin T and Swanson I 2004 N-impurity superstring spectra near the pp-wave limit *Nucl. Phys.* B **702** 86 (arXiv:hep-th/0407240)

[98] Ambjorn J, Janik R A and Kristjansen C 2006 Wrapping interactions and a new source of corrections to the spin-chain/string duality *Nucl. Phys.* B **736** 288 (arXiv:hep-th/0510171)

[99] Klose T and Zarembo K 2006 Bethe ansatz in stringy sigma models *J. Stat. Mech.* **0605** P006 (arXiv:hep-th/0603039)

[100] Arutyunov G and Frolov S 2006 On $AdS_5 \times S^5$ string S-matrix *Phys. Lett.* B **639** 378 (arXiv:hep-th/0604043)

[101] Hofman D M and Maldacena J M 2006 Giant magnons *J. Phys. A: Math. Gen.* **39** 13095 (arXiv:hep-th/0604135)

[102] Souriau J M 1970 *Structure des systèmes dynamiques* (Paris: Dunod)

[103] Kirillov A 1976 *Elements of the Theory of Representations* (translated by E Hewitt) (Berlin: Springer-Verlag) (English trans.)

[104] Arnold V I 1989 *Mathematical Methods of Classical Mechanics* (Berlin: Springer)

[105] Lukyanov S L and Shatashvili S L 1993 Free field representation for the classical limit of quantum Affine algebra *Phys. Lett.* B **298** 111 (arXiv:hep-th/9209130)

[106] Lukyanov S L 1995 Free field representation for massive integrable models *Commun. Math. Phys.* **167** 183 (arXiv:hep-th/9307196)

[107] Faddeev L D 1980 Quantum completely integral models of field theory *Sov. Sci. Rev.* C **1** 107

[108] Mintchev M, Ragoucy E, Sorba P and Zaugg P 1999 Yangian symmetry in the nonlinear Schroedinger hierarchy *J. Phys. A: Math. Gen.* **32** 5885 (arXiv:hep-th/9905105)

[109] Ragoucy E 2002 Vertex operators for quantum groups and application to integrable systems *J. Phys. A: Math. Gen.* **35** 7929 (arXiv:math/0108207)

[110] Bernard D and Leclair A 1991 Quantum group symmetries and nonlocal currents in 2-D QFT *Commun. Math. Phys.* **142** 99

[111] Bernard D and Leclair A 1993 The quantum double in integrable quantum field theory *Nucl. Phys.* B **399** 709 (arXiv:hep-th/9205064)

[112] Ahn C and Nepomechie R I 2008 The Zamolodchikov–Faddeev algebra for open strings attached to giant gravitons *J. High Energy Phys.* JHEP05(2008)059 (arXiv:0804.4036[hep-th])

[113] Beisert N and Klose T 2006 Long-range gl(n) integrable spin chains and plane-wave matrix theory *J. Stat. Mech.* **0607** P006 (arXiv:hep-th/0510124)

[114] Frolov S and Tseytlin A A 2003 Quantizing three-spin string solution in $AdS_5 \times S^5$ *J. High Energy Phys.* JHEP07(2003)016 (arXiv:hep-th/0306130)

[115] Frolov S and Tseytlin A A 2003 Rotating string solutions: AdS/CFT duality in non-supersymmetric sectors *Phys. Lett.* B **570** 96 (arXiv:hep-th/0306143)

[116] Arutyunov G, Frolov S, Russo J and Tseytlin A A 2003 Spinning strings in $AdS_5 \times S^5$ and integrable systems *Nucl. Phys.* B **671** 3 (arXiv:hep-th/0307191)

[117] Arutyunov G, Russo J and Tseytlin A A 2004 Spinning strings in $AdS_5 \times S^5$: new integrable system relations *Phys. Rev.* D **69** 086009 (arXiv:hep-th/0311004)

[118] Frolov S A, Park I Y and Tseytlin A A 2005 On one-loop correction to energy of spinning strings in S(5) *Phys. Rev.* D **71** 026006 (arXiv:hep-th/0408187)

[119] Park I Y, Tirziu A and Tseytlin A A 2005 Spinning strings in $AdS_5 \times S^5$: one-loop correction to energy in SL(2) sector *J. High Energy Phys.* JHEP03(2005)013 (arXiv:hep-th/0501203)

[120] Schafer-Nameki S 2006 Exact expressions for quantum corrections to spinning strings *Phys. Lett.* B **639** 571 (arXiv:hep-th/0602214)

[121] Beisert N, Minahan J A, Staudacher M and Zarembo K 2003 Stringing spins and spinning strings *J. High Energy Phys.* JHEP09(2003)010 (arXiv:hep-th/0306139)

[122] Beisert N, Frolov S, Staudacher M and Tseytlin A A 2003 Precision spectroscopy of AdS/CFT *J. High Energy Phys.* JHEP10(2003)037 (arXiv:hep-th/0308117)

[123] Beisert N, Tseytlin A A and Zarembo K 2005 Matching quantum strings to quantum spins: one-loop versus finite-size corrections *Nucl. Phys.* B **715** 190 (arXiv:hep-th/0502173)

[124] Freyhult L and Kristjansen C 2005 Finite size corrections to three-spin string duals *J. High Energy Phys.* JHEP05(2005)043 (arXiv:hep-th/0502122)

[125] Hernandez R, Lopez E, Perianez A and Sierra G 2005 Finite size effects in ferromagnetic spin chains and quantum corrections to classical strings *J. High Energy Phys.* JHEP06(2005)011 (arXiv:hep-th/0502188)

[126] Beisert N and Freyhult L 2005 Fluctuations and energy shifts in the Bethe ansatz *Phys. Lett.* B **622** 343 (arXiv:hep-th/0506243)

[127] Schafer-Nameki S, Zamaklar M and Zarembo K 2009 Quantum corrections to spinning strings in $AdS_5 \times S^5$ and Bethe ansatz: a comparative study *J. High Energy Phys.* JHEP09(2005)051 (arXiv:hep-th/0507189)

[128] Schafer-Nameki S and Zamaklar M 2005 Stringy sums and corrections to the quantum string Bethe ansatz *J. High Energy Phys.* JHEP10(2005)044 (arXiv:hep-th/0509096)

[129] Gromov N and Kazakov V 2006 Double scaling and finite size corrections in sl(2) spin chain *Nucl. Phys.* B **736** 199 (arXiv:hep-th/0510194)

[130] Gromov N and Vieira P 2008 Constructing the AdS/CFT dressing factor *Nucl. Phys.* B **790** 72 (arXiv:hep-th/0703266)

[131] Gromov N and Vieira P 2008 Complete 1-loop test of AdS/CFT *J. High Energy Phys.* JHEP04(2008)046 (arXiv:0709.3487[hep-th])

[132] Beisert N and Tseytlin A A 2005 On quantum corrections to spinning strings and Bethe equations *Phys. Lett.* B **629** 102 (arXiv:hep-th/0509084)

[133] Freyhult L and Kristjansen C 2006 A universality test of the quantum string Bethe ansatz *Phys. Lett.* B **638** 258 (arXiv:hep-th/0604069)

[134] Cachazo F, Spradlin M and Volovich A 2007 Four-loop cusp anomalous dimension from obstructions *Phys. Rev.* D **75** 105011 (arXiv:hep-th/0612309)

[135] Giangreco Marotta Puletti V, Klose T and Ohlsson Sax O 2008 Factorized world-sheet scattering in near-flat $AdS_5 \times S^5$ *Nucl. Phys.* B **792** 228 (arXiv:0707.2082[hep-th])

[136] Hentschel A, Plefka J and Sundin P 2007 Testing the nested light-cone Bethe equations of the $AdS_5 \times S^5$ superstring *J. High Energy Phys.* JHEP05(2007)021 (arXiv:hep-th/0703187)

[137] Gomez C and Hernandez R 2006 The magnon kinematics of the AdS/CFT correspondence *J. High Energy Phys.* JHEP11(2006)021 (arXiv:hep-th/0608029)

[138] Plefka J, Spill F and Torrielli A 2006 On the Hopf algebra structure of the AdS/CFT S-matrix *Phys. Rev.* D **74** 066008 (arXiv:hep-th/0608038)